BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 6, November 1979

RESEARCH ANNOUNCEMENTS

NONLINEAR SIMILARITY OF MATRICES

BY SYLVAIN E. CAPPELL[1] AND JULIUS L. SHANESON[1]

Works of Poincaré [7], de Rham [8], [9],[2] Reidemeister, Kuiper and Robbin [6], Sullivan, and Schultz [11] showed that in a large number of cases, various types of nonlinear similarity of real matrices with eigenvalues all of modulus one,[3] implies linear similarity of these matrices. In [4] we gave the first examples of matrices with eigenvalues of modulus one which are *nonlinearly similar* but, as they have different traces, are not linearly similar. Therefore the classification of such matrices up to nonlinear similarity is much different from what had been conjectured on the basis of the earlier results. This paper begins the systematic *classification up to nonlinear similarity of matrices with eigenvalues of modulus one* (and group representations) *which are not linearly similar.* This could be called the *topological canonical form problem for matrices*. For a large class of matrices (and group representations) we completely solve this problem.

Two real entried invertible $n \times n$ matrices A and B are *nonlinearly* or *topologically similar* if there is a homeomorphism $f\colon R^n \longrightarrow R^n$ with[4] $f(0) = 0$ and $fAf^{-1} = B$; here A and B are regarded as (linear) homeomorphisms of R^n.

For matrices with eigenvalues of modulus 1, and without roots of unity as eigenvalues, or with all eigenvalues which are roots of unity being sth roots of unity with $s = 1, 2, 3, 4$, or 6, Kuiper and Robbin showed that nonlinear similarity implies linear similarity. This is also known for matrices all of whose eigenvalues are primitive sth roots of unity, for a fixed s; see [6]. Dennis Sullivan, and Reinhard Schultz [11] showed nonlinear similarity implies similarity for matrices whose eigenvalues are p^s or $2p^s$ roots of unity, p an odd prime.

Received by the editors June 20, 1979.

AMS (MOS) subject classifications (1970). Primary 57E05, 57A15, 15A21; Secondary 54C05, 12A50, 34D05, 22A05, 15A18.

[1]Research partially supported by National Science Foundation Grants.

[2]For a detailed proof of de Rham's result and extensions to the P. L. case see Rothenberg [10].

[3]The classification of all matrices up to nonlinear similarity reduces to the case of matrices with eigenvalues of modulus 1, see Kuiper and Robbin [6], cf. [1].

[4]If the condition $f(0) = 0$ is dropped the equivalence classes of topologically similar matrices are not changed.

0002-9904/79/0000-0502/$02.00

For simplicity we consider first $n \times n$ matrices A for which the minimal polynomial has no multiple roots. This includes, in particular, all orthogonal matrices. Such matrices A are linearly similar if and only if they have the same characteristic polynomials, $p_A(t)$. Alternatively, they are determined up to linear similarity by their eigenvalues, the roots of $p_A(t)$ counted with multiplicity.

Let $[x]$ denote as usual the largest integer less than or equal to the real number x.

THEOREM. *Let A and B be a real $n \times n$ matrices with all eigenvalues of A of modulus one and with the minimal polynomial of A having no multiple roots (for example: A an orthogonal matrix). Then A and B are nonlinearly similar if B has all eigenvalues of modulus one with its minimal polynomial having no multiple roots and the characteristic polynomials have real factorizations*

$$p_A(t) = k(t)h(t), \qquad p_B(t) = k(t)h(-t) \tag{1}$$

where $h(t) = 1$ if $k(-1) \neq 0$ and where

$$h(t) = \prod_{q>1} h_q(t),$$

degree $h_q(t)$ divisible by 4, with the roots of $h_q(t)$ primitive $4q$th roots of unity and, counting these roots with multiplicity, for each $q > 1$,

$$(q + n)\,(\deg(h_q(t))/4) + \sum_{\substack{x \text{ a root of } h_q \\ \operatorname{Im}(x)>0}} [((-x^q \ln x)/\pi)\cdot n]$$

is even for n any integer with $1 \leqslant n \leqslant 2q - 1$.

Conversely, this condition is also necessary for B to be nonlinearly similar to A provided that each integer s, for which A has an eigenvalue which is a primitive sth *root of unity, satisfies*

(i) *s is an odd prime-power or twice an odd prime-power or less than* 8, *or*

(ii) *A has less than 2 eigenvalues*[5] *which are primitive sth roots of unity, or*

(iii) *A has less than 3 eigenvalues which are nonprimitive sth roots of unity.*

REMARK. In view of (ii) and (iii) one can add: *or*

(iv) *A has less than 6 eigenvalues which are sth roots of unity.*

For $n = 2$, Poincaré showed by defining rotation numbers that homogeneous nonlinear similarity implies linear similarity. In [4] we gave a counterexample in a higher dimension. However, by using the full force of the above theorem, we show

[5]Of course, eigenvalues are counted with multiplicity.

COROLLARY. *For all $n \times n$ matrices with all eigenvalues of modulus one, when $n \leqslant 6$, nonlinear similarity is equivalent to linear similarity.*

Note that (1) implies, in particular, that A^2 and B^2 are linearly similar. In some cases, the homeomorphisms used in constructing the topological similarity of the not linearly similar matrices can be chosen to be diffeomorphisms on almost all of R^n.

The criteria (1) for nonlinear similarity is implied by a simpler condition (1').

$$(1') \qquad p_A(t) = k(t)\Big(\prod_{q>1} h_q(t)\Big)^2, \qquad p_B(t) = k(t)\Big(\prod_{q>1} h_q(-t)\Big)^2$$

where all $h_q(t) = 1$ if $k(-1) \neq 0$ and the roots of $h_q(t)$ are primitive $4q$th roots of unity and the degree of each $h_q(t)$ is divisible by 4.

Moreover, condition (1') is equivalent to (1) if A does not have -1 as an eigenvalue, or if for all nontempered integers $4q$ greater than 80, A has less than 8 eigenvalues[5] which are primitive $4q$th roots of unity. To define tempered, consider the function f from the integers modulo $4q$ to the integers modulo 2, $f\colon Z_{4q} \to Z_2$, given by $f(x) = 1$ for $x = 1, 2, \ldots, (2q-1)$ and $f(x) = 0$ for $x = 2q, (2q+1), \ldots, 4q$. Call $4q$ a *tempered* number if the functions f_a, where $f_a(x) = f(ax)$, for $a \in Z_{4q}$, a prime to q and $a \equiv 1$ (modulo 4), satisfy only the linear identities over Z_2 which are sums of the obvious identities

$$f_a + f_{(2q+(-1)^q a)} = f_1 + f_{(2q+(-1)^q)} \text{ (modulo 2)}.$$

Among the tempered numbers are 2^r and $4q$, q a Fermat prime, all $4q < 84$, and other classes of numbers.

These theorems give results on topological equivalence of different cyclic group representations. In [4] we announced the first example of this phenomenon (for each Z_{4q}, $q > 1$). Here we will give a sample of our classification results for group representations. Let t^a denote the underlying 2-dimensional real representation of the complex representation which sends the generator of Z_{4q} to $e^{2\pi a/4q}$. The real representations of Z_{4q} are well known to be sums of t^a, $0 < a < 2q$ together with the trivial and nontrivial one-dimensional representations, δ_{+1} and δ_{-1}, of Z_{4q}.

Let Z_{4q}^* denote the elements of Z_{4q} which have multiplicative inverses.

COROLLARY. *A representation γ of Z_{4q} is topologically equivalent to*

$$\delta_{-1} + \sum_{\substack{a \in Z_{4q}^* \\ 0<a<2q}} c_a t^a,$$

nonnegative integers if, and also when $4q$ is tempered only if,

$$\gamma = \delta_{-1} + \sum_{\substack{a \in Z_{4q}^* \\ 0<a<2q}} d_a t^a, \qquad d_a \geq 0,$$

where for each a, $c_a - d_a = d_{2q-a} - c_{2q-a}$ *is even, and* $\Sigma_{0<a<q} (c_a - d_a)$ *is divisible by* 4.

There are even more exotic examples of topologically equivalent linear representations of Z_{4q} when $4q$ is not tempered. The first example of this is for 84.

EXAMPLE. *The representation of* Z_{84}, $\delta_{-1} + t^5 + t^{13} + t^{17} + t^{23} + t^{31} + t^{41}$ *is topologically equivalent to* $\delta_{-1} + t^1 + t^{11} + t^{19} + t^{25} + t^{29} + t^{37}$.

In [4] we also gave an example of homogeneous topological similarity of different representations. Further discussion of this will appear in [5].

SKETCH OF PROOF. These results follow from classifying up to topological similarity the linear representations of Z_{4q} which are free actions on R^n with a line deleted and fixed point free on $R^n - 0$.

The quotient space of such a representation on Euclidean space is obtained by compactifying, at one end, $R \times Y$ where Y is the one-point compactification of a nontrivial line-bundle over a lens space. By studying the classification of these spaces $R \times Y$ and relating it to invariants of lens spaces, we show that finding a topological similarity is the same as solving a certain multiplicative equation involving the corresponding Reidemeister torsions. Further algebraic reductions show that the existence of a solution to these equations is equivalent to the condition (1) above.

REFERENCES

1. V. I. Arnold, *Ordinary differential equations*, M.I.T. Press, Cambridge, Mass., 1973.
2. S. E. Cappell and Julius L. Shaneson, *Pseudo-free group actions*. I, Proc. 1978 Aarhus Topology Conf. (to appear).
3. ———, *Which groups have pseudo-free actions on spheres* (to appear).
4. ———, *Linear algebra and topology*, Bull. Amer. Math. Soc. (N.S.) 1 (1979), 685–687.
5. ———, *Linear representations which are topologically the same* (to appear).
6. N. Kuiper and J. W. Robbins, *Topological classification of linear endomorphisms*, Invent. Math. 19 (1973), 83–106.
7. H. Poincaré, *Sur les courbes définies par les équations différentielles*, Oeuvres de H. Poincaré, Vol. I 1928, Gauthier Villars, Paris.
8. G. de Rham, *Reidemeister's torsion invariant and rotations of* S^n, Internat. Conf. on Differential Analysis, Oxford Univ. Press, Cambridge, 1964, pp. 27–36.
9. G. de Rham, S. Maumary and M. Kervaire, *Torsion et type simple d'homotopie*, Lecture Notes in Math., vol. 48, Springer-Verlag, Berlin, New York, 1967.
10. M. Rothenberg, *Torsion invariants and finite transformation groups*, Proc. Sympos. Pure Math., vol. 32, Amer. Math. Soc., Providence, R. I., pp. 267–312, 1978.
11. R. Schultz, *On the topological classification of linear representations* (to appear).

COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, NEW YORK 10012