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BOOK REVIEWS

Introduction to categories, homological algebra and sheaf cohomology, by Jan R. Strooker, Cambridge Univ. Press, Cambridge, Great Britain, 1978, ix + 246 pp.

Category theory is not just a branch of mathematics; it is, in some sense, the language of all mathematics. As Strooker so aptly puts it: "categories are there to make different topics more transparent by revealing common underlying patterns." Before elaborating on this point, let me begin by reviewing some of the fundamental concepts of category theory for the uninitiated.

An *oriented graph* consists of two classes, the class of *arrows* and the class of *objects*, together with two mappings between them, called *source* and *target*.

$$\text{arrows} \underset{\text{target}}{\overset{\text{source}}{\rightrightarrows}} \text{objects}$$

Instead of saying that source $(f) = A$ and target $(f) = B$, we write more briefly $f: A \to B$.

A *category* is an oriented graph in which for every object A there is an *identity* arrow $1_A: A \to A$ and which is endowed with a *composition* of arrows as follows

$$\frac{f: A \to B \quad g: B \to C}{gf: A \to C}.$$

Moreover, the following equations are postulated:

$$1_B f = f, \qquad f 1_A = f, \qquad (hg)f = h(gf),$$

where $h: C \to D$.

Objects of interest to mathematicians usually flock together in categories. Thus we have the category **Top** of topological spaces, whose objects are topological spaces and whose arrows are continuous functions. We also have the category **Grp** of groups, whose objects are groups and whose arrows are homomorphisms. Last but not least, there is the category **Sets** of sets and mappings between them.

Closer inspection shows that the objects of mathematics may themselves be categories. Thus a topological space X may be viewed as a category whose objects are the open subsets of X and whose arrows are the inclusion mappings between open subsets. More generally, any preordered set may be regarded as a category: elements are objects, and there is at most one arrow $x \to y$ for any pair of objects, precisely one when $x \leq y$. Also a group, or even a monoid, may be considered as a category with only one object whose arrows are the elements. Finally, a set may be looked upon as a *discrete* category in which there are no arrows other than identities.

A modern Pythagoras might have said: "all things are categories." To this a modern Heraclitus might have replied that it is not the categories that are important but the functors between them.

A *functor* $T: \mathcal{A} \to \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$ consists of two functions, both denoted by T, from the class of objects of $\mathcal{A}$ to that of $\mathcal{B}$ and from the class of arrows of $\mathcal{A}$ to that of $\mathcal{B}$, subject to the conditions that

$$\frac{f: A \to B}{T(f): T(A) \to T(B)}, \qquad T(1_A) = 1_{T(A)}, \qquad T(gf) = T(g)T(f).$$

For example, any topological group G determines a functor U_G: $\mathbf{Top}^{\mathrm{op}} \to \mathbf{Grp}$. Here $\mathcal{A}^{\mathrm{op}}$ denotes the *opposite* of the category $\mathcal{A}$: It has the same objects as $\mathcal{A}$, but its arrows are reversed. $U_G(X)$ is the group of all continuous functions from the topological space X to the underlying space $|G|$ of G, and $U_G(f)(h) = hf$ for all f: $Y \to X$ and h: $X \to |G|$ in **Top**. Also, a functor between preordered sets is an order preserving mapping and a functor between monoids is a homomorphism.

Not surprisingly, categories themselves are the objects of a category **Cat** whose arrows are functors. For set-theoretical reasons, one usually insists that the objects of **Cat** are *small* categories, that is, categories whose classes of objects and arrows are sets.

Given categories $\mathcal{A}$ and $\mathcal{B}$, we may consider the functors $\mathcal{A} \to \mathcal{B}$ as objects of a new category $(\mathcal{A}, \mathcal{B})$ whose arrows t: $S \to T$ are *natural transformations*, that is, mappings t from the class of objects of $\mathcal{A}$ to that of arrows of $\mathcal{B}$ such that $t(A)$: $S(A) \to T(A)$ for each object A of $\mathcal{A}$, subject to the equation

$$T(f)t(A) = t(A')S(f)$$

for all arrows f: $A \to A'$ in $\mathcal{A}$, as illustrated by the following "commutative" diagram

$$\begin{array}{ccc} S(A) & \overset{t(A)}{\to} & T(A) \\ S(f)\downarrow & & \downarrow T(f) \\ S(A') & \underset{t(A')}{\to} & T(A') \end{array}$$

It has been said that categories were invented by Eilenberg and Mac Lane in order to explain what natural transformations are, these having been observed to occur in mathematics previously, the usual example being the isomorphism between a finite-dimensional vector space and its double dual.

Our modern Heraclitus would not have been content to emphasize the importance of functors, he would have pointed out that they frequently occur in pairs. F: $\mathcal{A} \to \mathcal{B}$ is called *left adjoint* to U: $\mathcal{B} \to \mathcal{A}$ if there are natural transformations η: $1_{\mathcal{A}} \to UF$ and ε: $FU \to 1_{\mathcal{B}}$ such that the composite natural transformations

$$U \overset{\eta U}{\to} UFU \overset{U\varepsilon}{\to} U, \qquad F \overset{F\eta}{\to} FUF \overset{\varepsilon F}{\to} F$$

are identities, that is to say,

$$U(\varepsilon(B))\eta(U(B)) = 1_{U(B)}, \qquad \varepsilon(F(A))F(\eta(A)) = 1_{F(A)}$$

for all objects A of $\mathcal{A}$ and B of $\mathcal{B}$.

Adjoint functors generalize Galois correspondences between preordered sets. Examples abound in mathematics. The functor F which assigns to every set X the free group $F(X)$ generated by X is left adjoint to the "forgetful" functor U: **Grp** $\to$ **Sets** which assigns to each group B its underlying set $U(B)$. The functor U_G: $\mathbf{Top}^{\mathrm{op}} \to \mathbf{Grp}$ associated with a topological group G met earlier has a left adjoint F_G which assigns to each group A the topological space of all homomorphisms from A to the underlying group of G. In fact, every pair of adjoint functors $\mathbf{Top}^{\mathrm{op}} \rightleftarrows \mathbf{Grp}$ comes from a topological group in this way.

A pair of adjoint functors $\mathcal{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{B}$ is an *equivalence* of categories if $\eta(A)$: $A \to UF(A)$ and $\varepsilon(B)$: $FU(B) \to B$ are isomorphisms for all objects A of $\mathcal{A}$ and B of $\mathcal{B}$.

To illustrate this strict notion of equivalence, let $\mathcal{A}^{\mathrm{op}}$ be the category of compact (Hausdoff) Abelian groups, $\mathcal{B}$ the category of abstract Abelian groups. Given any compact Abelian group A, $F(A)$ is the group of all continuous homomorphisms of A into the compact group $\mathbf{R}/\mathbf{Z}$. Given any abstract Abelian group B, $U(B)$ is the group of all homomorphisms of B into the underlying abstract group of $\mathbf{R}/\mathbf{Z}$, with a topology induced by $\mathbf{R}/\mathbf{Z}$. The fact that $\eta(A)$ and $\varepsilon(B)$ are isomorphisms is the celebrated Pontrjagin duality.

In any case, a pair of adjoint functors $\mathcal{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{B}$ gives rise to an equivalence between certain full subcategories $\mathcal{A}_0$ of $\mathcal{A}$ and $\mathcal{B}_0$ of $\mathcal{B}$. (*Full* means that if A and A' are in $\mathcal{A}_0$ then so is any arrow $A \to A'$ of $\mathcal{A}$.) Here $\mathcal{A}_0$ consists of all objects A of $\mathcal{A}$ for which $\eta(A)$ is an isomorphism and $\mathcal{B}_0$ consists of all objects B of $\mathcal{B}$ for which $\varepsilon(B)$ is an isomorphism.

$$\begin{array}{ccc} \mathcal{A} & \underset{U}{\overset{F}{\rightleftarrows}} & \mathcal{B} \\ \text{inclusion} \uparrow & & \uparrow \text{inclusion} \\ \mathcal{A}_0 & \simeq & \mathcal{B}_0 \end{array}$$

Heraclitus would have called the equivalence $\mathcal{A}_0 \simeq \mathcal{B}_0$ the "unity of opposites".

Under fairly general circumstances it will happen that ηUF is an isomorphism, which implies that εFU is also an isomorphism. In that case the inclusion functor $\mathcal{A}_0 \to \mathcal{A}$ has a left adjoint and one calls $\mathcal{A}_0$ a full *reflective* subcategory of $\mathcal{A}$. The inclusion functor $\mathcal{B}_0 \to \mathcal{B}$ then also has a right adjoint and $\mathcal{B}_0$ becomes a full *coreflective* subcategory of $\mathcal{B}$.

For example, let $\mathcal{A}$ be the category of rings and $\mathcal{B}$ the category $\mathbf{Top}^{\mathrm{op}}$. Consider the two-element ring $\mathbf{Z}/(2)$ as a topological ring with the discrete topology. Let $U(B)$ be the ring of continuous functions from the topological space B to the underlying discrete space of $\mathbf{Z}/(2)$, and let $F(A)$ be the topological space of homomorphisms of A into the underlying ring of $\mathbf{Z}/(2)$. Then F is left adjoint to U, the induced equivalence being between the category $\mathcal{A}_0$ of Boolean rings and the category $\mathcal{B}_0$, where $\mathcal{B}_0^{\mathrm{op}}$ is the category of zero-dimensional compact Hausdorff spaces. This is the famous Stone duality. Moreover, in this example, Boolean rings form a reflective subcategory of the category of rings and zero-dimensional compact Hausdorff spaces form a reflective subcategory of **Top**.

In a similar fashion, we obtain a pair of adjoint functors between the category $\mathcal{A}$ of Banach algebras and the category $\mathcal{B}$ = $\mathbf{Top}^{op}$ mediated by the Banach algebra of complex numbers. This time $\mathcal{A}_0$ is the category of commutative C^*-algebras and $\mathcal{B}_0^{op}$ is the category of compact Hausdorff spaces. Moreover, the latter is a reflective subcategory of **Top**, and the left adjoint of the inclusion functor is the well-known Stone-Čech compactification.

Having expressed a personal view of category theory and its applications, let me acknowledge a philosophical debt to Bill Lawvere and point out that details of the above examples have been elaborated in collaboration with Basil Rattray. It is now time to turn to the book under review.

Strooker does not wish to lose sight of the origin and purpose of category theory. His book is rich in illustrative examples taken from different parts of mathematics. In particular, he evidently set himself the aim of preparing the reader for a study of the cohomology of sheaves. In his own words: "Sheaves and their cohomology are important tools in such diverse mathematical disciplines as algebraic topology, theory of analytic functions, algebraic geometry and others. Though there are differences in their use in these different areas, they have an underlying pattern in common which is best expressed in categorical language."

Given the author's aim, the plan of the book is determined. Since sheaves of modules form at best an Abelian category (see below), homological algebra must be developed not just for module categories, but for Abelian categories in general. Let us take a detailed look at the four chapters of the book.

Chapter 1 deals with general concepts of category theory more or less along the lines sketched above. However, it should be pointed out that the author's definition of a category is more traditional than the above. In fact, he assumes that, for all objects A and B, the class of arrows $A \to B$ is a set. This has the disadvantage that for $(\mathcal{A}, \mathcal{B})$ to be a category he must assume that $\mathcal{A}$ is small, that is, has only a set of arrows. It has the advantage that one has a canonical functor Hom: $\mathcal{A}^{op} \times \mathcal{A} \to$ **Sets**. I liked the careful treatment of adjoint functors; but I did not like the fact that functors $\mathcal{A}^{op} \to \mathcal{B}$ are sometimes described as "contravariant functors $\mathcal{A} \to \mathcal{B}$", albeit that this is a traditional point of view.

The first chapter soon concentrates on the notion of limit. Given a functor T: $\mathcal{I} \to \mathcal{A}$, one may look at the category of *lower bounds* of T: its objects are pairs (A, t) where A is an object of $\mathcal{A}$ and t is a natural transformation to T from the functor with constant value A; its arrows $(A, t) \to (A', t')$ are arrows a: $A \to A'$ such that $t'(I)a = t(I)$ for all objects I of $\mathcal{I}$. It may happen that the category of lower bounds of T has a *terminal* object (A_0, t_0), by which is meant that from every other object (A, t) there is a unique arrow to (A_0, t_0). Then (A_0, t_0) is usually called the *limit* of T. For obvious reasons (and following the reviewer's past practice), the author calls (A_0, t_0) the "infimum" of T. The more usual term "limit" comes from a special case: the *inverse limit* of an inverse family of objects of $\mathcal{A}$ is the limit of T: $\mathcal{I} \to \mathcal{A}$, denoted by $\varprojlim T$, when $\mathcal{I}$ is a downward directed set, here regarded as a category. Inverse limits in $\mathcal{A}^{op}$ are also called *direct limits* and denoted by $\varinjlim T$.

Of special interest are two other cases of the notion of limit. At one extreme, $\mathcal{I}$ may be discrete, that is, have no arrows other than identities. Then lim T is the *product* of all $T(I)$, I ranging over the objects of $\mathcal{I}$. At the other extreme, we have the case when $\mathcal{I}$ is the category $I \overset{i}{\underset{j}{\rightrightarrows}} J$. Then lim T is the *equalizer* of $T(i)$ and $T(j)$ in $\mathcal{A}$. It turns out that arbitrary limits may be constructed from products and equalizers. $\mathcal{A}$ is called *complete* if limits of T: $\mathcal{I} \to \mathcal{A}$ exists whenever $\mathcal{I}$ is small, that is, when $\mathcal{A}$ has small products and equalizers.

The author gives an up-to-date version of Freyd's adjoint functor theorem, which states conditions under which a limit preserving functor from a complete category $\mathcal{B}$ into some other category possesses a left adjoint. These conditions are trivially satisfied when $\mathcal{B}$ is complete and small, as the author remarks. It ought to be pointed out that small complete categories must be preordered sets, hence complete lattices, as has been observed by Freyd; but I could not find this information in the present text.

A highpoint is the author's treatment of Kan extensions. If $\mathcal{C}$ is a complete category and I: $\mathcal{A} \to \mathcal{B}$ is any functor, the induced functor $(I, \mathcal{C})$: $(\mathcal{B}, \mathcal{C}) \to (A, \mathcal{C})$, where $(I, \mathcal{C})(G) = GI$ for all G: $\mathcal{B} \to \mathcal{C}$, has a left adjoint S. By the *Kan extension* of F: $\mathcal{A} \to \mathcal{C}$ along I is meant the functor $S(F)$: $\mathcal{B} \to \mathcal{C}$. This concept turns out to be very useful, as we shall see.

Chapter 2 deals mainly with additive and Abelian categories, but begins with monomorphisms and epimorphisms, whose discussion has been wisely deferred until this point. A *monomorphism m* is an arrow with the cancellation property $mf = mg \Rightarrow f = g$, and *epimorphisms* are defined dually, they are just monomorphisms in the opposite category.

In familiar concrete categories monomorphisms are injective mappings, but epimorphisms are not necessarily surjective. Thus, in the category of Hausdorff spaces, a continuous function e: $A \to B$ is an epimorphism if and only if $e(A)$ is dense in B.

A category $\mathcal{A}$ is called *additive* if Hom(A, B) is an Abelian group, for each pair of objects A and B, and if composition of arrows is bilinear. If there is a terminal object in an additive category, it is also *initial*, that is, terminal in the opposite category; it is then called a *null* object and denoted by 0. In an additive category all finite products are also *coproducts*, that is, products in the opposite category. In an additive category the equalizer of f: $A \to B$ and 0: $A \to B$ is called the *kernel* of f; its importance arises from the fact that the equalizer of f, g: $A \to B$ is the kernel of $f - g$. *Cokernels* are kernels in the opposite category.

Of special interest are *Abelian* categories, which may be defined as additive categories with null objects, kernels and cokernels, such that every monomorphism is a kernel, every epimorphism is a cokernel and every arrow is composed of an epimorphism, called the *coimage*, followed by a monomorphism, called the *image*. There are many other ways of describing Abelian categories, and the author–in his own words–"does not resist some pretty juggling of axioms", following Puppe.

In an Abelian category one may define "exact sequences", a pair of morphisms $A \to B \to C$ being *exact* if the image of $A \to B$ is the kernel of $B \to C$. One may also engage in the activity known as "diagram chasing" to prove results about exact sequences and commutative diagrams. Crucial is the following result, known as the *snake lemma*: given the diagram,

$$
\begin{array}{ccccccccccccc}
 & & & & & & & & 0 & & & & \\
 & & & & & & & & \downarrow & & & & \\
 & & & & & & K & \to & K'' & \to & X & \to & 0 \\
 & & & & & & \downarrow & 5 & \downarrow & 6 & \downarrow & & \\
 & & & & A' & \to & A & \to & A'' & \to & 0 & & \\
 & & & & \downarrow & 3 & \downarrow & 4 & \downarrow & & & & \\
 & & 0 & \to & B' & \to & B & \to & B'' & & & & \\
 & & \downarrow & 1 & \downarrow & 2 & \downarrow & & & & & & \\
0 & \to & Y & \to & C' & \to & C & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & & & 0 & & & & & & & &
\end{array}
$$

in which all rows and all columns are exact and all squares are commutative, then $X \cong Y$. In view of the so-called "connecting" arrow $K'' \to X \cong Y \to C'$, we then obtain an exact sequence $K \to K'' \to C' \to C$ of considerable importance in homological algebra.

To prove the snake lemma, or other such results, one has a number of options.

(1) One can prove these results for module categories and then use the Mitchell embedding theorem to deduce them for arbitrary Abelian categories.

(2) One can construct and utilize binary relations for Abelian categories, as is done in Mac Lane's "Homology" for module categories.

(3) Without explicitly mentioning relations, one can mimic them with the help of broken arrows.

(4) One can use the technique of chasing squares (see below).

The author choses (3). He eschews the embedding theorem and the theory of relations altogether and proves the snake lemma for Abelian categories directly by means of ingenious auxiliary diagrams. The reviewer cannot resist the temptation to push his pet method (4), which is based on the following *two-square lemma*

$$
\begin{array}{ccccc}
A & \to & B & \to & C \\
\downarrow & \mathrm{I} & \downarrow & \mathrm{II} & \downarrow \\
D & \to & E & \to & F
\end{array}
$$

Assuming that both squares are commutative and that both rows are exact, then

$$\frac{\operatorname{Im}(D \to E) \cap \operatorname{Im}(E \to E)}{\operatorname{Im}(A \to E)} \cong \frac{\operatorname{Ker}(B \to F)}{\operatorname{Ker}(B \to C) + \operatorname{Ker}(B \to E)}.$$

This may be summarized by writing $\operatorname{Im} \mathrm{I} \cong \operatorname{Ker} \mathrm{II}$. Here U/V is understood in the sense that $0 \to V \to U \to U/V \to 0$ is exact. This lemma holds not only in Abelian categories, but in more general kinds of categories, including the category of groups. Its general proof is due to Leicht.

To deduce the snake lemma, one just argues thus

$$\operatorname{Im} 1 \cong \operatorname{Ker} 2 \cong \operatorname{Im} 3 \cong \operatorname{Ker} 4 \cong \operatorname{Im} 5 \cong \operatorname{Ker} 6.$$

To compute Im 1 and Ker 6, we look at the auxiliary diagrams

$$\begin{array}{ccccccccccc} & & 0 & \rightarrow & B' & & 0 & \rightarrow & X & & \\ & & \downarrow & 1 & \downarrow & & \downarrow & 7 & \downarrow & & \\ 0 & \rightarrow & Y & \rightarrow & C' & & K'' & \rightarrow & X & \rightarrow & 0 \\ & & \downarrow & 0 & \downarrow & & \downarrow & 6 & \downarrow & & \\ & & Y & \rightarrow & 0 & & A'' & \rightarrow & 0 & & \end{array}$$

and note that

$$\text{Im } 1 \cong \text{Ker } 0 \cong Y, \quad \text{Ker } 6 \cong \text{Im } 7 \cong X.$$

Abelian categories of special interest are Grothendieck categories. A *Grothendieck* category is a cocomplete Abelian category with a generator G and exact direct limits. Here $\mathcal{A}$ is called *cocomplete* if $\mathcal{A}^{op}$ is complete. An object G of $\mathcal{A}$ is called a *generator* if for all $f \neq g$: $A \rightarrow B$ in $\mathcal{A}$, one can find h: $G \rightarrow A$ such that $fh \neq gh$. To say that $\mathcal{A}$ has *exact direct limits* means that, for every upward directed set $\mathcal{I}$, the functor $\varinjlim: (\mathcal{I}, \mathcal{A}) \rightarrow \mathcal{A}$ (which is left adjoint to the constant functor) preserves exact sequences.

Grothendieck categories are essentially characterized by the Gabriel-Popescu theorem. This celebrated theorem asserts: if R is the ring of endomorphisms of the generator G of the Grothendieck category $\mathcal{A}$, the functor $U = \text{Hom}(G, -)$: $\mathcal{A} \rightarrow \text{Mod } R$ is full and faithful and has a left adjoint F which preserves monomorphisms, hence preserves exact sequences. To say that U is *faithful* means that, if $f \neq g$: $A \rightarrow B$ in $\mathcal{A}$, then $U(f) \neq U(g)$; to say that U is *full* means that every arrow $U(A) \rightarrow U(B)$ in Mod R has the form $U(f)$ for some f: $A \rightarrow B$ in $\mathcal{A}$. The theorem may be interpreted as saying that $\mathcal{A}$ is equivalent to a full subcategory of Mod R which consists of the "torsionfree-divisible" modules with respect to some injective module.

I shall take a moment to explain what this means. A "torsion theory" in Mod R is determined by an *injective* module I, that is, a module I so that, for every monomorphism m: $B \rightarrow A$, every homomorphism f: $B \rightarrow I$ can be extended to some f': $A \rightarrow I$ such that $f'm = f$. The *torsionfree* modules with respect to I are those modules which admit a monomorphism into some power I^X. The *divisible* modules with respect to I are those modules D for which $E(D)/D$ is torsionfree, where $E(D)$ is the *injective hull* of D, that is, the minimal injective extension of D. The modules which are both torsionfree and divisible with respect to I form a full reflective subcategory of Mod R which is a Grothendieck category. The Gabriel-Popescu theorem then asserts that, up to equivalence, every Grothendieck category is of this form.

The original proof of the Gabriel-Popescu theorem makes extensive use of torsion theories. The author gives an ingenious elementary proof that the functor F: Mod $R \rightarrow \mathcal{A}$ preserves monomorphisms. He first considers a monomorphism $M \rightarrow L$ where M is finitely generated and L is free, he then drops the condition that M is finitely generated, and finally drops the condition that L is free. This proof is ascribed to Harada in **[16]**.

The importance of the Gabriel-Popescu theorem lies in the fact that it allows one to infer many properties of $\mathcal{A}$ from those of Mod R, for example, that $\mathcal{A}$ is complete, has injective hulls and possesses an injective cogenerator.

Chapter 3 deals with homological algebra in an arbitrary Abelian category $\mathcal{A}$. If A and C are objects of $\mathcal{A}$, Ext(C, A) consists of all equivalence classes of exact sequences $0 \to A \to B \to C \to 0$ in $\mathcal{A}$. It is traditional to call B an *extension* of C by A, although to a ring theorist B is an extension of A and not of C. Under mild assumptions on $\mathcal{A}$, $\mathrm{Ext}(C, A)$ will be a set rather than merely a class. This is the case not only when $\mathcal{A}$ is small, but also when $\mathcal{A}$ has enough projectives, that is, for every object A there is an epimorphism $P \to A$ with P *projective*, that is, injective in $\mathcal{A}^{\mathrm{op}}$. It is also the case, dually, if $\mathcal{A}$ has enough injectives. One then obtains a functor Ext from $\mathcal{A}^{\mathrm{op}} \times \mathcal{A}$ to **Sets** or to the category **Ab** of Abelian groups.

Given an exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, one obtains an exact sequence of Abelian groups

$$0 \to \mathrm{Hom}(A'', C) \to \mathrm{Hom}(A, C) \to \mathrm{Hom}(A', C)$$
$$\to \mathrm{Ext}(A'', C) \to \mathrm{Ext}(A, C) \to \mathrm{Ext}(A', C).$$

One may wish to continue this exact sequence to the right, using $\mathrm{Ext}^n(A, C)$, where $\mathrm{Ext}^0(A, C) = \mathrm{Hom}(A, C)$ and $\mathrm{Ext}^1(A, C) = \mathrm{Ext}(A, C)$. Indeed, $\mathrm{Ext}^n(C, A)$ may be defined with the help of exact sequences

$$0 \to A \to B_1 \cdots \to B_n \to C \to 0,$$

just as in the case $n = 1$.

According to Cartan-Eilenberg there are two ways of generalizing the above setup from $\mathrm{Ext}(-, C)$ to arbitrary additive functors from an Abelian category to the category **Ab** of Abelian groups. "The slow but elementary iterative procedure leads to the notion of 'satellite functors'. The faster, homological method using resolutions leads to the 'derived functors'. In most important cases both procedures yield identical results".

Strooker's discussion of derived functors is traditional and I shall say nothing about it. However, his treatment of satellites is unusual and follows a suggestion by Gabriel. He constructs the additive category $\mathcal{A}^e$ with objects (A, m), where A is in $\mathcal{A}$ and $m \in \mathbf{N}$. Arrows $(A, m) \to (B, n)$ are elements of $\mathrm{Ext}^{n-m}(A, B)$, and composition of arrows is an associative operation discovered by Yoneda. Suppose $\mathcal{A}$ is small and $\mathcal{B}$ is cocomplete as well as additive, and consider the functor $(I, \mathcal{B})$: $(\mathcal{A}^e, \mathcal{B}) \to (\mathcal{A}, \mathcal{B})$ induced by the obvious inclusion I: $\mathcal{A} \to \mathcal{A}^e$. It has a left adjoint S, which assigns to each F: $\mathcal{A} \to \mathcal{B}$ its Kan extension $S(F)$: $\mathcal{A}^e \to \mathcal{B}$. The additive functors from $\mathcal{A}^e$ to $\mathcal{B}$ are in one-to-one correspondence with "connected" sequences of additive functors from $\mathcal{A}$ to $\mathcal{B}$. The sequence of functors corresponding to $S(F)$ is called the sequence of *satellites* of the additive functor F.

Chapter 4 finally deals with sheaves and their cohomology. The best way to introduce sheaves is yet another example of Heraclitean unity of opposites. Let $\mathcal{A}$ be the functor category $(X^{\mathrm{op}}, \mathbf{Sets})$, also called the category of *presheaves* over X, where X is a given topological space, here regarded as a category. Let $\mathcal{B} = \mathbf{Top}/X$ be the category of *spaces over* X, whose objects are continuous functions p: $Y \to X$, Y in **Top**, and whose arrows from p: $Y \to X$ to p': $Y' \to X$ are continuous functions f: $Y \to Y'$ such that $p'f = p$. Let U: $\mathcal{B} \to \mathcal{A}$ assigns to each p: $Y \to X$ the functor $U(p)$: $X^{\mathrm{op}} \to \mathbf{Sets}$ such that, for each open subset V of X, $U(p)(V)$ is the set of continuous sections, that is, functions f: $V \to Y$ with $pf =$ the inclusion $V \to X$. It is not difficult to see that U has a left adjoint F. Indeed, for each presheaf A, $F(A)$ may be described simple-mindedly as the colimit of all open subsets V of X indexed by all elements $a \in A(V)$.

In this situation we have an equivalence of full subcategories $\mathfrak{B}_0$ of $\mathfrak{B}$ and $\mathfrak{A}_0$ of $\mathfrak{A}$. Here $\mathfrak{A}_0$ is the category of *sheaves* on X and $\mathfrak{B}_0$ the category of *local homeomorphisms* $p: Y \to X$, also called "espaces étalés", and the left adjoint to the inclusion $\mathfrak{A}_0 \to \mathfrak{A}$ is the so-called *sheafification* functor.

Strooker treats sheaves essentially by the method just outlined, except that he takes $\mathfrak{B} = \mathfrak{B}_0$ to start with and that, like other writers on the subject, he constructs $F(A)$ as the disjoint union of *stalks* $p^{-1}(x) = \varinjlim \{A(V) | x \in X\}$, over all $x \in X$, endowed with a certain topology.

From sheaves of sets he passes to sheaves of modules. These form an Abelian category, to which the apparatus of homological algebra now applies. In particular, the right derived functors of the global section functor from sheaves of R-modules to Mod R are studied as forming a cohomology theory in the sense of Eilenberg-Steenrod. The last few pages are devoted to special kinds of sheaves that go under the picturesque names "flabby", "soft", and "fine".

The book under review began as a set of lecture notes at the University of Utrecht ten years ago. It was lovingly translated into English by C. J. Penning, who is also credited for having suggested many revisions. The author asserts quite modestly that the book was intended as a textbook for students and not as a monograph for mature mathematicians. In this reviewer's opinion, even many of the latter could benefit from reading it.

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Convexity in the theory of lattice gases, by Robert B. Israel, Princeton Series in Physics, Princeton Univ. Press, Princeton, N. J., 1979, LXXXV + 168 pp., $16.50 (cloth).

Lattice gases? This sounds very much like physics. And what have they to do with convexity? The mathematician may be pardoned if he is puzzled, but he could't do better than to look into this book if he wants to find out what this is all about. Lattice gases are certain mathematical models that occur in statistical mechanics. Statistical mechanics was created a near-century ago by J. Willard Gibbs, who conceived the idea of a general explanation for the laws of thermodynamics. It was also Gibbs who in two pioneering papers, rather neglected ever since, suggested that the proper general formulation of the laws of thermodynamics may be made in terms of certain functions, called thermodynamics potentials, which characterize the physical systems considered, and whose *convexity* is the mathematical expression of the stability of states of thermal equilibrium. Our book under review is actually two books in one; the first is an introductory essay by Arthur Wightman, which contains the historical motivation, an exposition of the Gibbsian ideas, the significance of convexity of the thermodynamic potentials, as well as a brief review of the formalism of statistical mechanics as left to us by Gibbs. This is far more than an introduction, and it alone is worth the price of the book. The reader is advised to come back to it from time to time, when studying the more technical proofs of Israel's chapters, to gain motivation, deepen understanding, and appreciate interconnections.

On to the technicalities. First, definitions. A lattice gas is a mathematical system determined by five things, ν, Ω_0, μ_0, Ω and $\mathfrak{B}$. ν is a positive integer, the "dimension". Ω_0 is a compact Hausdorff space, frequently just a finite set. μ_0 is a distinguished natural normalized measure on Ω_0, e.g. Haar measure if Ω_0 is a group, uniform surface measure if Ω_0 is a sphere, normalized counting measure if Ω_0 is finite. With $\mathbf{Z}$ the set of integers, write $L = \mathbf{Z}^\nu$ (the "lattice"), and think of a copy of (Ω_0, μ_0) attached to each point ("lattice site") of L. Ω is defined as a closed, translation-invariant (under the natural action of the additive group L) subspace of the compact space Ω_0^L; thus a point ("configuration") $\omega \in \Omega$ is a function $L \to \Omega_0$ assigning a "coordinate" $\omega_x \in \Omega_0$ to each $x \in L$. A typical example is $\Omega_0 = \{0, 1\}$, $\Omega = \{\omega \in \Omega_0^L\colon\ \omega_x\omega_y = 0$