parameters. {The actual analyticity statement is more sophisticated since the interactions are parametrized by an infinite dimensional Banach space.} The proofs in Chapters 1–5 are elegant and complete but sometimes rather demanding of the reader.

In Chapter 6 the formalism of the first four chapters is partially extended to compact metrizable spaces with a $\mathbf{Z}^\nu$-action. An interesting wedding between the statistical mechanical formalism and topological dynamics is achieved. In Chapter 7 the richer formalism of Chapter 5 is extended to certain $\mathbf{Z}$-actions on Smale spaces. Most detailed proofs in the last two chapters are omitted but complete references are given. Exercises, some of them quite difficult, are given at the end of each chapter. There are also complete bibliographical notes at the end of each chapter.

This is a beautiful but austere book. It is very much in the spirit of the Bourbaki treatise. We must compare this impression with the statement of the editor in the general preface to this Encyclopedia. It states: "Clarity of exposition, *accessibility to the nonspecialist* (italics added), and a thorough bibliography are required of each author." If a person can learn a subject for the first time by reading Bourbaki, then perhaps that person can learn the statistical mechanics of lattice systems by reading this book. In this reviewer's opinion most people will most profitably read Bourbaki and/or this book at the culmination of the learning process not at the beginning.

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Modular forms and functions, by Robert A. Rankin, Cambridge University Press, Cambridge, London, New York, Melbourne, 1977, xiii + 384 pp., $34.00.

Modular functions and Dirichlet series in number theory, by Tom M. Apostol, Graduate Texts in Mathematics, Number 41, Springer-Verlag, New York-Heidelberg-Berlin, 1976, x + 198 pp., $14.80.

My first actual conversation with Mordell took place early in the 1960's, when we were introduced (by L. C. Young, I believe) in the lounge of the Mathematics Research Center in Madison, Wisconsin. Always interested in the work of young mathematicians–a phrase that applied to me then–Mordell asked about my research interests. To my answer he responded with surprise (possibly feigned, it occurred to me later; in any event the point is the same), saying in effect–I don't recall the exact words–"modular functions? I thought that was all settled years ago!"

That no mathematician, not even a Mordell in jest, could respond that way today is a measure of the extraordinary resurgence of interest that the field has enjoyed during the past ten years. More tangible indications are close at hand. In a review published in this Bulletin recently (March, 1976), I was able to mention, in addition to the book under review, seven automorphic/modular (one complex variable) books that had appeared in English, beginning in 1962, the year of publication of Gunning's brief (but influential) *Lectures on modular forms*. And that list easily could have been doubled in length had I made some effort to be comprehensive.

In the few intervening years, the level of activity in this field has not abated, and, indeed, it may well have increased. The research interest continues as well to be reflected in the publication of books, some intended as an introduction for the beginner, others–stressing recent developments and an advanced viewpoint–more appropriate for experts and aspirants to that lofty station. Of the two books under review here, Apostol's belongs to the first category, Rankin's to the second. Both books restrict themselves to "modular" (as opposed to the more general "automorphic") functions in a single complex variable, and–happily in my view–each is the product of a mathematical sensibility shaped largely by number-theoretic influences.

The modular group $\Gamma(1)$ and its invariants, the modular functions and modular forms, arise quite naturally in a variety of mathematical settings–equivalence theory over Z of binary quadratic forms with rational coefficients and the theory of elliptic functions are examples that come to mind immediately–typically with profound consequences. $\Gamma(1)$ is the matrix group $SL(2, Z)$-determinant 1; in the theory of modular functions it should be viewed also as the group of linear fractional transformations $\tau \to (a\tau + b)/(c\tau + d)$, with $a, b, c, d \in Z$ and $ad - bc = 1$. (Here, as elsewhere in this review, τ is a complex variable with values in the upper half-plane $\mathcal{H}$.) Beyond $\Gamma(1)$ itself, certain of its subgroups, the congruence groups, play an important role in applications to number theory and they figure prominently as well both in Apostol's exposition and in Rankin's.

If q is an integer ≥ 2, define $\Gamma(q)$, the *principal congruence group of level* q, to be that subgroup of $\Gamma(1)$ whose elements V satisfy the congruence condition $V \equiv \pm I \pmod{q}$. (I is the 2×2 identity matrix and the congruence is element-wise.) A *congruence subgroup* Γ of $\Gamma(1)$ is one that contains $\Gamma(q)$ for some q. Congruence groups of particular interest are the subgroups $\Gamma_0(q)$ of $\Gamma(1)$ defined by the restriction $c \equiv 0 \pmod{q}$, where again q is an integer ≥ 2. While every congruence subgroup has finite index in $\Gamma(1)$ (since $\Gamma(q)$ has finite index), the converse is false: subgroups (indeed, infinite classes of subgroups) of finite index which are not congruence groups have long been known. (This fact sets off strikingly the deep theorem of Bass, Lazard, and Serre (1964): every subgroup of finite index in $SL(n, Z)$ *is* in fact a congruence group, when $n \geq 3$.)

To any subgroup Γ of $\Gamma(1)$–of finite or infinite index–one can attach, though not uniquely, a fundamental domain, a subset $\mathfrak{F}$ of $\mathcal{H}$ which contains a single point from each orbit $\{V\tau_0 | V \in \Gamma\}$ (τ_0 fixed in $\mathcal{H}$) of Γ. Since such a set may be topologically bizarre (it may, for example, be totally disconnected), we impose additional restrictions upon $\mathfrak{F}$, ordinarily that it is connected and that the boundary has Lebesgue measure zero. For $\Gamma(1)$ no problem arises in this connection, since one can take as a fundamental domain the set $\{\tau \in \mathcal{H} \mid |\tau| > 1 \text{ and } |\text{Re } \tau| < \frac{1}{2}\}$, with "half" of the boundary points adjoined. For subgroups of finite index in $\Gamma(1)$, and for many other discrete subgroups of $SL(2, R)$ as well, an equally satisfactory construction exists for $\mathcal{F}$.

Suppose Γ is a subgroup of finite index in $\Gamma(1)$. If f is a meromorphic function on $\mathcal{H}$, call f a *modular function on* Γ, provided f is invariant under Γ (that is to say, $f \circ V = f$ for all $V \in \Gamma$) and if in addition f satisfies a growth condition–to be described below–at the parabolic cusps of $\mathcal{F}$, those points (necessarily finite in number) in which the topological closure of $\mathcal{F}$ relative to the Riemann sphere meets the extended real line. If $k \in Z$ and f, meromorphic on $\mathcal{H}$, satisfies the functional equation

$$f(V\tau) = (c\tau + d)^k f(\tau), \quad V = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma, \tag{$*$}$$

we say f is a *modular form of weight k with respect to* Γ, provided again that f satisfies the appropriate growth condition at the cusps. Note that a modular function is simply a modular form of weight $k = 0$ and that, with f a modular function, f' is a modular form of weight 2 on the same subgroup, since $f(V\tau) = f(\tau)$ implies $(c\tau + d)^{-2} f'(V\tau) = f'(\tau)$. (This is not to say, however, that every modular form of weight 2 is the derivative of a modular function, just as there are meromorphic differentials on a Riemann surface which are not exact.)

Every subgroup Γ of finite index in $\Gamma(1)$ contains a translation, and in fact the translations of Γ form a cyclic subgroup. Suppose $\tau \to \tau + \lambda$, $\lambda > 0$, generates that subgroup. The functional equation ($*$), with $V\tau = \tau + \lambda$, implies $f(\tau + \lambda) = f(\tau)$, so that one may expect a Fourier expansion (actually a Laurent expansion in the variable $e^{2\pi i\tau/\lambda}$) of the form

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\tau/\lambda}, \tag{$**$}$$

valid when Im τ is large. In fact, to derive ($**$) we must impose the additional condition that there is a half-plane Im $\tau > y_0$ ($y_0 > 0$) containing no poles of f; then ($**$) is valid in this same half-plane. The growth condition needed in the definition of modular form is the assumption that ($**$) is finite to the left:

$$f(\tau) = \sum_{n > \mu} a_n e^{2\pi i n\tau/\lambda}, \quad \mu \in Z.$$

At each of the finite cusps of $\mathcal{F}$ there is a similar condition, the same in conception but technically more complicated. The key point is that for each finite parabolic cusp P of $\mathcal{F}$ there is in Γ a cyclic subgroup of transformations fixing P, with parabolic (trace $= \pm 2$) generator. This fact implies the validity near P of an expansion having the general form ($**$), in an exponential variable determined by the generator; we demand of a modular form that the expansion be finite to the left for each P. If, in addition, f is holomorphic in $\mathcal{H}$ and no cusp expansion contains a term with $n < 0$, f is called an *entire modular form*. When f is an entire form in which all of the expansions contain only terms with $n > 0$, f is said to be a *cusp form*. The vector spaces M_k^0 of cusp forms and M_k of entire forms, of fixed weight k, are finite dimensional when Γ is a subgroup of finite index in $\Gamma(1)$, and for similar discrete subgroups of $SL(2, R)$. (When Γ is not contained in $\Gamma(1)$, the functions and forms are called "automorphic" rather than "modular". Apostol adopts the unusual practice of using the term "automorphic" for forms on a proper subgroup of $\Gamma(1)$.)

Modular functions and Dirichlet series in number theory is a sequel to Apostol's earlier work, *Introduction to analytic number theory*. Both volumes, the author tells us, "evolved from a course . . . offered at the California Institute of Technology during the last 25 years." By any reasonable standard of measurement, twenty-five years is a long gestation period for a book, yet the rewards in careful selection of topics, smooth, accurate statement of theorems and definitions and a generous selection of good exercises for the reader (somewhat unusual in works on this subject), amply justify the wait. Reflecting its genesis and evolution, the book is in fact ideally suited to the classroom and to seminars at the graduate-student level.

The final two chapters, on Kronecker's approximation theorem and Bohr's equivalence theorem for general Dirichlet series, respectively, relate–but only indirectly–to the first six, which comprise an introduction to the theory of modular functions. The connecting thread is the theory of Dirichlet series. (One of the two appealing applications of Kronecker's theorem given in Chapter 7 is to the calculation of $\inf|\zeta(s)|$ and $\sup|\zeta(s)|$ on a fixed vertical line $\operatorname{Re} s = \sigma_0 > 1$, where $\zeta(s)$ is the Riemann zeta function.) Long favored by number theorists, these functions have enjoyed renewed attention in very recent years, as mathematicians have increasingly come to appreciate the importance of Hecke's discovery (mid 1930's) of the correspondence between an entire modular form $f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau}$ on $\Gamma(1)$ and the Mellin transform of $f(\tau) - a_0$, the Dirichlet series $\Phi_f(s) = (2\pi)^{-s}\Gamma(s) \sum_{n=1}^{\infty} a_n n^{-s}$. (Here s is restricted to a suitable right-half plane.) Apostol gives a simple, readable account of that part of Hecke's work in which the elegant functional equation

$$(-1)^{k/2}(2\pi)^{s-k}\Gamma(k - s)\Phi_f(k - s) = (2\pi)^{-s}\Gamma(s)\Phi_f(s),$$

analogous to that satisfied by $\zeta(s)$, is derived from the transformation formula $f(-1/\tau) = \tau^k f(\tau)$, the equation (*) with V replaced by the modular transformation $T\tau = -1/\tau$. But, like Rankin (and a number of other authors as well), Apostol omits the proof of the converse, despite its intrinsic interest and even though it involves little more than a reversal of the steps in the derivation of the direct result.

Apostol succeeds in introducing the classical modular functions rapidly and painlessly by first presenting the Weierstrass development of the elliptic functions. The differential equation satisfied by the Weierstrass $\wp$ function gives rise directly to the Eisenstein series $g_2(\tau)$, $g_3(\tau)$, the modular forms of weights 4 and 6, respectively, defined by $g_2(\tau) = 60\ \Sigma(m\tau + n)^{-4}$, $g_3(\tau) = 140\ \Sigma(m\tau + n)^{-6}$. (Here τ is the ratio of the fundamental periods of $\wp$ and the summation is over all pairs of integers other than $(m, n) = (0, 0)$.) From g_2 and g_3 it is but a short step to the discriminant function $\Delta(\tau)$, a modular form of weight 12, and to the basic modular invariant (weight 0) $J(\tau) = g_2^3(\tau)/\Delta(\tau)$. The author carries out all of this without wasted motion; by p. 20 he has derived the expression for the Fourier coefficients of g_2 and g_3 in terms of divisor functions and proved the integrality of the coefficients of $(2\pi)^{-12}\Delta(\tau)$ and $12^3J(\tau)$.

Efficiency and clarity mark the further developments as well. Included among these is Picard's (little) theorem, whose proof Apostol establishes by applying the mapping properties of $J(\tau)$, rather than the more usual method involving $\lambda(\tau)$, the fundamental invariant of the principal congruence subgroup of level 2. This procedure has the obvious advantage that it does not require a prior discussion of modular functions on subgroups, a subject which is not introduced until two chapters later in the book, and then only for the congruence subgroup $\Gamma_0(q)$.

In giving the definition of a modular function on $\Gamma_0(q)$ Apostol omits the condition of left-finiteness ordinarily imposed upon the Fourier expansion at a finite parabolic point (to guarantee that such a function is meromorphic on the compactification of the Riemann surface $\mathcal{H}/\Gamma_0(q)$), though he includes the corresponding–technically simpler–condition at ∞. Whether this is an oversight or done intentionally to streamline the exposition, there is a happy ending. For the modular functions on $\Gamma_0(q)$ occur here in two contexts only, and in neither one does the omission render the development incorrect or misleading. The first of these is the theorem that a modular (Apostol calls it "automorphic") function on $\Gamma_0(p)$, p a prime, which is also bounded in $\mathcal{H}$, is constant; here the assumption of boundedness subsumes the condition of left-finiteness at all of the parabolic points, including ∞. This fundamental theorem, incidentally, is not restricted to $\Gamma_0(p)$; indeed, it holds for all finitely generated Fuchsian groups of the first kind, including the (very large) class of subgroups of finite index in $\Gamma(1)$. The proof that Apostol gives is valid–virtually without alteration–for such subgroups of $\Gamma(1)$.

The second context is the explicit construction of modular functions belonging to $\Gamma_0(q)$, in terms of invariant functions on the full group $\Gamma(1)$ or from the modular form $\Delta(\tau)$, of weight 12 on $\Gamma(1)$. ($\Delta(q\tau)/\Delta(\tau)$ is an instance of the latter.) In either case, the left-finiteness results from the construction. The principle of using functions on the full group $\Gamma(1)$ to obtain functions on a proper subgroup has long been known and applied to good effect. It is illustrated quite simply by observing that if $f(\tau)$ belongs to (i.e., is invariant on) $\Gamma(1)$, then $f(q\tau)$, $q \in Z^+$, belongs to $\Gamma_0(q)$ and to no larger subgroup of $\Gamma(1)$. The functions which Apostol wants to obtain have a more complex rule of formation, based upon the *Hecke operators* (which originated with Mordell):

$$T(n)f(\tau) = n^{-1} \sum_{\substack{ad=n \\ b(\mathrm{mod}\, d) \\ d>0}} f\left(\frac{a\tau + b}{d}\right), \quad n \in Z^+.$$

It is a simple matter to check that $T(n)$ preserves the space of functions with period 1, somewhat more difficult to prove the significant fact that $T(n)$ preserves the space of modular functions on $\Gamma(1)$. A slight modification of $T(n)$ can be made to yield Hecke operators $T_k(n)$ which preserve the space of modular forms of integral weight k on $\Gamma(1)$. The resulting operators preserve both M_k and M_k^0 as well.

When p is a prime, $T(p)$ takes on the simplified form $T(p)f(\tau) = p^{-1}f(p\tau) + \Sigma_{\lambda=0}^{p-1} f\{(\tau + \lambda)/p\}$. Apostol studies the functions $f_p(\tau) = T(p)f(\tau) - p^{-1}f(p\tau) = \Sigma_{\lambda=0}^{p-1} f\{(\tau + \lambda)/p\}$, with f a modular function on $\Gamma(1)$. Since $T(p)f(\tau)$ belongs to $\Gamma(1)$ and $f(p\tau)$ to $\Gamma_0(p)$, $f_p(\tau)$ belongs to $\Gamma_0(p)$–and to no larger subgroup of $\Gamma(1)$. The author applies this construction to obtain congruences modulo 2^{11}, 3^5, 5^2 and 7 for the coefficients of $j(\tau) = 12^3 J(\tau)$, by deriving an expression for $j_p(\tau)$ as a polynomial, with integer coefficients, in a suitable power of $\Delta(p\tau)/\Delta(\tau)$.

The theory of modular and automorphic forms can be extended to include nonintegral weights, and indeed the generalization to arbitrary real weight has been carried out in detail, largely by Hans Petersson during the 1930's. (Petersson's important, wide-ranging contributions include the introduction in 1939 of the inner product–bearing his name–which gives M_k^0 the structure of a Hilbert space.) However, modular forms of half-integral weights arising as theta functions, and related series or products, received a good deal of study in the 19th century (beginning with the work of Jacobi in the first third of the century, long before the formulation of a theory of modular forms *per se*), motivated for the most part by their deep connections with elliptic functions and number-theoretic functions.

The discussion of arbitrary real weights requires a generalization of the notion of a modular form that creates some complications, but no essential new difficulties. Specifically, if k is an arbitrary real number, replace the equation (*) by

$$f(V\tau) = \nu(V)(c\tau + d)^k f(\tau), \quad V = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma, \qquad (***)$$

where the branch of $(c\tau + d)^k$ is specified by the condition $-\pi \leqslant \arg(c\tau + d) < \pi$, and $|\nu(V)| = 1$ (hence ν is independent of τ) for each $V \in \Gamma$. If f then is meromorphic in $\mathcal{H}$ and satisfies the appropriate left-finiteness condition at each parabolic cusp of $\mathcal{F}$, it is called a *modular form of weight k, with multiplier system ν*. Of course, if $k \in Z$ and ν is the trivial multiplier system ($\nu(V) = 1$ for all $V \in \Gamma$) in (***), f is a modular form in the sense of (*), but even with $k \in Z$, ν need not be trivial, and in fact the greater generality introduced with the multiplier systems enriches the theory significantly for integral weight.

The Dedekind eta function

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi in\tau})$$

is an important example of a modular form with nonintegral weight, and Apostol devotes to it an entire chapter with two distinct derivations of the deep-lying functional equation

$$\eta(-1/\tau) = e^{-\pi i/4}\tau^{1/2}\eta(\tau).$$

($\eta(\tau)$ is a modular form of weight $\frac{1}{2}$ with a complicated multiplier system involving the number-theoretically fascinating Dedekind sums.) One of these is Siegel's elegant proof based upon the residue theorem. The second, less widely known and more general, rests upon Shô Iseki's transformation formula (1957), which amply deserves the attention Apostol gives it here. The author ultimately applies the transformation properties of $\eta(\tau)$ in the "circle method" to derive Rademacher's remarkable convergent series representation of the partition function $p(n)$, which arises as the coefficient in the Fourier expansion of the modular form $\eta(\tau)^{-1}$. The reader may well be grateful for the excellent "plan of the proof" with which the presentation begins. Intended as a guide through the intricate, potentially bewildering labyrinth of details inherent in the circle method, this outline is only one striking example of Apostol's talent for lucidity and organization.

Insofar as the theory of entire (bounded at ∞) modular forms on $\Gamma(1)$ is concerned, the heart of the book is Chapter 6, where one finds most of the standard topics which belong in a good introductory work: the formula for the number of zeros of an entire form; the proof of representability of an arbitrary entire form as a polynomial in g_2 and g_3 (a subject of renewed interest in the past several years)–and the resultant elementary calculation of the dimensions of the vector spaces M_k and M_k^0 of entire and cusp (vanishing at ∞) forms; the Hecke operators, $T_k(n)$, that preserve M_k and M_k^0; multiplicative properties and order of magnitude estimates of the Fourier coefficients of modular forms. The chapter closes with the discussion–previously mentioned–of the Hecke correspondence between modular forms and Dirichlet series with functional equations.

If Apostol's is a teacher's book, Rankin's is primarily a scholar's. This is not to say that *Modular forms and functions* is unsuited to the classroom. In fact, it can be used with graduate students, but success may depend upon their prior familiarity with an introductory book on the level of Apostol's. On the other hand, more experienced mathematicians who intend to learn the subject in some depth would be well advised to study Rankin's treatment carefully. For, as much as any recent exposition of modular functions, this book succeeds in getting near the research frontier, and in some instances even reaches it–no small feat in this theory. Only someone of Rankin's stature as a research mathematician and experience in the classroom could aspire to such an accomplishment in a self-contained work–beginning with first principles.

The book, indeed, is informed throughout by insight and perspective of a level attained only through long years of persistent creative effort. This is nowhere more evident than in Chapters 8 and 9, which deal with a number of linear operators acting on the space of modular forms of fixed integer weight k, on the principal congruence group $\Gamma(q)$. Rankin's achieves coherence and a measure of unity in a deep, far-ranging discussion of these operators (including the difficult generalization of the Hecke operators to modular forms on $\Gamma(q)$), their invariant subspaces and consequences for the Fourier coefficients of modular forms on $\Gamma(q)$. He integrates material extending from the early work of Hecke to the recent influential results of Atkin-Lehner (1970) concerning Hecke operators and newforms on $\Gamma_0(q)$, including as well a good bit of his own original work. The achievement is not without its price, however: the intrinsic difficulties of the material are unavoidably reflected in the style of exposition and in a formidable array of notation. The beginner is certain to be daunted–or, at least, discouraged–by these chapters, which nevertheless will handsomely reward the serious attention of those who are prepared for them.

In their restriction to modular forms of integral weight and trivial (identically 1) multiplier systems, Chapters 8 and 9 deviate from the general practice of the book, which for the most part treats modular forms of arbitrary real weight with corresponding arbitrary multiplier system. In addition, throughout the first half of the book the group in question is an arbitrary subgroup of finite index in $\Gamma(1)$, while only in the latter half, where rather explicit results are sought, is the group restricted either to $\Gamma(1)$ itself or to a congruence subgroup. As might be expected, there are annoying complexities which cannot be avoided in discussing modular forms of arbitrary real weight, yet this generality (or, at least, half-integral weight, for which the complications are virtually the same) is essential to the most significant number-theoretic applications. On the one hand, Rankin could have done without the whole of Chapter 3–and a good deal of additional text besides–had he been content with forms of integral weight and trivial multiplier system. On the other hand, a consequence of this would have been the necessary reduction of his (rather full) treatment of the important Fourier coefficients of $\vartheta^s(\tau)$ to a mere shadow of the one presently found in Chapter 7.

The very definition of the classical theta function

$$\vartheta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}, \quad \operatorname{Im} \tau > 0,$$

reveals it as a natural tool in attacking problems relating to sums of squares. Like the Dedekind eta function, $\vartheta(\tau)$ is a modular form of weight $1/2$, but on a subgroup Γ_ϑ of index 3 in $\Gamma(1)$ rather than the full group. Its multiplier system v_ϑ, related to that of $\eta(\tau)$, is expressible, alternatively, in terms of Gaussian sums or the Jacobi quadratic symbol. Furthermore, $v_\vartheta^8 \equiv 1$. When s is a positive integer, the Fourier coefficient $r_s(n)$ of $\vartheta^s(\tau)$–a modular form of weight $s/2$ and multiplier system v_ϑ^s on Γ_ϑ–counts the number of ways to represent n as a sum of s integral squares. Rankin's development of Eisenstein series of arbitrary real weight, together with his calculation (without use of the Riemann-Roch theorem) of the dimension of the vector space of cusp forms on Γ_ϑ of real weight $s/2$ and multiplier system v_ϑ^s, enable him to get an explicit formula for $r_s(n)$ when $4 < s \leqslant 8$. Applying Hecke's estimate for the coefficients of cusp forms, he obtains in addition a good asymptotic formula for $r_s(n)$ when $s > 8$. Had Rankin developed the theory for integral weight and trivial multiplier system only, this unified presentation of results for arbitrary real s–previously known but nowhere gathered together–would necessarily have been replaced by one restricted to integral $s \equiv 0 \pmod 8$, with dramatically reduced impact.

Another highlight of Rankin's exposition is his detailed, careful discussion of the Petersson theory of parabolic Poincaré series, which furnish a convenient mode of representing all modular forms, holomorphic in $\mathcal{H}$, of real weight $k > 2$, on a subgroup of finite index in $\Gamma(1)$. (Petersson actually developed the theory for any finitely generated Fuchsian group of the first kind, but the case of subgroups of $\Gamma(1)$ exposes all of its salient features.) From the properties of the Poincaré series relative to the Petersson inner product, it follows that a finite number of them–the Eisenstein series are a special case–form a basis for the space of entire modular forms of fixed weight $k > 2$. Rankin uses the Petersson inner product once again to derive interesting symmetry relations between the Fourier coefficients of two Poincaré series attached to the same group, weight and multiplier system. Probing more deeply into the structure of the Fourier coefficients, he obtains their explicit infinite series representations, first discovered by Petersson. These formulae are derived from Rankin's useful generalization of the Lipschitz summation formula (not identified as such), and they involve Bessel functions as well as generalized Kloosterman sums, exponential sums which in special cases have a number-theoretic structure.

In all of these developments the restriction to $k > 2$ is essential. Imposed to guarantee absolute convergence of the Poincaré series, it can be relaxed somewhat, but only with great difficulty. Rankin's application of the Hecke limiting process to construct modular forms of weight 2 and trivial multiplier system on $\Gamma(q)$ only begins to suggest the nature and magnitude of the problems that must be overcome when $k \leqslant 2$. The Kloosterman sums corresponding to the case $k = 2$ with trivial multiplier system on $\Gamma(q)$ have number-theoretic properties from which follows a nontrivial estimate on their order of magnitude. Rankin devotes a good deal of attention to deriving this growth estimate, which is essential to the successful application of Hecke's method.

The Poincaré series, as noted, can be used to represent any modular form, when $k > 2$. The author gives, in addition, several special representation theorems valid for the case of the full modular group $\Gamma(1)$. These are: every entire modular form on $\Gamma(1)$ of even integral weight and trivial multiplier system is a polynomial in the Eisenstein series g_2 and g_3 (proved also by Apostol); every modular form on $\Gamma(1)$ of even weight and trivial multiplier system (allowing poles in $\mathcal{H}$ and at ∞) is a *rational* function of g_2 and g_3. There are other interesting special results given for $\Gamma(1)$, among them the modular equation for $J(\tau)$, identities involving Eisenstein series of various (even integral) weights and a proof that such Eisenstein series are eigenforms simultaneously for all the Hecke operators $T_k(n)$ connected with $\Gamma(1)$.

In *Modular forms and functions*, Rankin has given us a densely written, often difficult book, yet one of great importance for advanced students and experts. To the main body of exposition, which weaves together standard topics and a good deal of material that is hard–if not impossible–to find elsewhere, Rankin adds a bibliography of some 170 items, and an historical addendum to each chapter, which reveal the depth of his scholarship. Above all, however–in these pages one continually feels the guiding presence of a first-class mathematical mind.

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