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RICHARD GRAFF

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Permanents, by Henryk Minc, Encyclopedia of Mathematics and its Applications (Gian-Carlo Rota, Editor), Volume 6, Addison-Wesley, Reading, Mass., 1978, xviii + 205 pp., $21.50.

The year 1979 can be regarded as the 20th anniversary of the *theory* of the permanent function. True, permanents were introduced in 1812 by Binet [2] and Cauchy [9], and several identities, usually involving determinants as well, were obtained in the 19th century by some ten other mathematicians including Cayley and Muir. Indeed it was Sir Thomas Muir [30] who in 1882 coined the term 'permanent' for the following function defined on $n \times n$ matrices $A = [a_{ij}]$:

$$\operatorname{per} A = \sum_{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

where the summation extends over all $n!$ permutations σ of $\{1, \ldots, n\}$. True, in 1903 Muirhead [31] obtained the following beautiful result. Let $c = (c_1, \ldots, c_n)$ be a positive n-tuple, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta =$

$(\beta_1, \ldots, \beta_n)$ be fixed n-tuples of nonnegative integers. Define $n \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ by $a_{ij} = c_i^{\alpha_j}$ and $b_{ij} = c_i^{\beta_j}$ $(i, j = 1, \ldots, n)$. Then always $\operatorname{per} A \leqslant \operatorname{per} B$ if and only if α is majorized by β, in which case equality occurs if and only if either $\alpha = \beta$ or $c_1 = \cdots = c_n$. Here α is *majorized* by β means the following: If $(\alpha_1^*, \ldots, \alpha_n^*)$ and $(\beta_1^*, \ldots, \beta_n^*)$ denote the n-tuples α and β rearranged in nonincreasing order, then $\alpha_1^* + \cdots + \alpha_k^* \leqslant \beta_1^* + \cdots + \beta_k^*$ $(k = 1, \ldots, n)$ with equality for $k = n$. Muirhead's theorem was extended by Hardy, Littlewood and Pólya [**19**] to apply to nonnegative n-tuples α and β.

Ten years after Muirhead proved his result, Pólya [**33**] showed that for $n \geqslant 3$ there is no uniform way to affix $\pm$ signs to the entries of an $n \times n$ matrix so that the determinant of the resulting matrix equals the permanent of the original matrix. And then Schur [**37**] in 1918 proved the lovely theorem that for a positive semidefinite hermitian matrix A, $\det A \leqslant \operatorname{per} A$, with equality if and only if A is a diagonal matrix or has a zero row. Surely the theorems of Muirhead and Schur and the intriguing implications of Pólya's negative finding (is there some linear transformation on the linear space of $n \times n$ matrices which will 'convert' the permanent into the determinant?) would be an ample justification for further study of this scalar-valued matrix function. But in 1926 van der Waerden [**40**] asked the following question and then quietly walked away. Call a matrix *doubly stochastic* if its entries are nonnegative, and all of its row and column sums equal 1. What is the minimum value of the permanent of an $n \times n$ doubly stochastic matrix? Apparently no one knew, nor did anyone rush to find out. Skipping a few details, we arrive at 1959.

In 1959 Marvin Marcus and Morris Newman [**26**] attempted to answer van der Waerden's question. Let Ω_n denote the convex polytope of all $n \times n$ doubly stochastic matrices and let J_n denote the $n \times n$ matrix in Ω_n all of whose entries equal $1/n$. Then Marcus and Newman proved that if A is an $n \times n$ doubly stochastic matrix all of whose entries are positive such that

$$\operatorname{per} A = \min\{\operatorname{per} B\colon B \in \Omega_n\},$$

then $A = J_n$ and thus $\operatorname{per} A = n!/n^n$. In other words, if the minimum of the permanent is achieved in the interior of Ω_n, then it is achieved uniquely there at J_n. But why shouldn't the minimum occur on the boundary of Ω_n? The widely conjectured answer to van der Waerden's question is $n!/n^n$ where the only matrix in Ω_n whose permanent equals $n!/n^n$ is J_n. More to come on the van der Waerden conjecture later. In addition to the Marcus and Newman paper another significant paper was published in 1959 by Brenner [**5**]. This paper proved some theorems about permanents but almost incidentally. Call an $m \times n$ complex matrix $(m \leqslant n)$ $A = [a_{ij}]$ *diagonally dominant* if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for $i = 1, \ldots, m$. Then Brenner obtained a collection of inequalities relating the determinant of a principal submatrix of a diagonally dominant matrix A with the determinants of the other submatrices of A on the same set of rows. As a corollary one obtains the classical result that the determinant of a square diagonally dominant matrix is not zero. A referee observed that the crucial property of determinants used in the proofs (the Laplace expansion by a set of rows) held for permanents as well, so that it followed in particular that the permanent of a square diagonally dominant matrix is not zero. Brenner's paper went almost unnoticed until 1967 when it was observed by Brenner and Brualdi [6] that it could be used to easily settle a conjecture of Marcus and Minc [24]. The papers of Marcus and Newman and of Brenner marked the beginning of the efforts of many mathematicians to settle the van der Waerden conjecture and, in general, to understand the permanent function.

Permanents follow the definition of determinants with the simplification that the $\pm$ sign which appears in front of the $n!$ terms in the determinant becomes a $+$ sign for each term in the case of the permanent. A simplification in the definition which however results in tremendous complications when one tries to evaluate permanents. Of course, hardly ever does one evaluate the determinant of a matrix by computing the terms one by one. Rather one uses elementary row operations and the behavior of the determinant under them to reduce the matrix to another whose determinant is obvious. But permanents are not well behaved under row operations (since the permanent of a matrix with two equal rows need not be zero), and as a result it is usually not possible to accurately compute the permanent of a matrix unless the matrix has a special form. But why would one want to evaluate the permanent of a matrix anyway? Not for the same reasons one often needs to evaluate a determinant. One important reason is based on the following. First note that the permanent can be defined for any $m \times n$ matrix $A = [a_{ij}]$ with $m \leqslant n$ by

$$\operatorname{per} A = \sum_{\sigma} a_{1\sigma(1)} \cdots a_{m\sigma(m)}$$

where the summation extends over all injections σ from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$. Now let $X_1, \ldots, X_m$ be a family of subsets of a set $S = \{x_1, \ldots, x_n\}$ of n elements. A sequence $s_1, \ldots, s_m$ of m distinct elements of S is a *system of distinct representatives* for $X_1, \ldots, X_m$ provided $s_i \in X_i$ $(i = 1, \ldots, m)$. Let $A = [a_{ij}]$ be the $m \times n$ incidence matrix defined by $a_{ij} = 1$ if $x_j \in X_i$ and $a_{ij} = 0$ if $x_j \notin X_i$. For this matrix A, $a_{1\sigma(1)} \cdots a_{m\sigma(m)} = 1$ if $x_{\sigma(1)}, \ldots, x_{\sigma(m)}$ is a system of distinct representatives for $X_1, \ldots, X_m$, and 0 otherwise. Hence the permanent of A equals the number of systems of distinct representatives for $X_1, \ldots, X_m$. Thus, for instance, the classical'problème des ménages' (the number of ways n couples can be placed at a circular table so that men and women sit in alternate places and no husband sits next to his wife) has as solution the number $2n!$ times the permanent of the $n \times n$ matrix all of whose entries equal 1 except for those in positions $(1, 1), \ldots, (n, n), (1, 2), \ldots, (n-1, n), (n, 1)$ which are 0. Likewise the important dimer problem of statistical physics [17] (the number of ways to disect an n-dimensional parallelepiped into dimers) is easily reduced to evaluating the permanent of a matrix with many 0's. For $n = 2$ Percus [32] showed that this matrix A has a complex companion $\tilde{A}$ such that $\operatorname{per} A = |\det \tilde{A}|$. For $n \geqslant 3$, Hammersley et al. [18] showed that no such mate $\tilde{A}$ can be found even if one allows quaternion entries. It follows that good estimates for the permanent of matrices can be valuable. The most efficient known method for computing a general permanent was derived by Ryser [34] from the inclusion-exclusion principle.

Since evaluation of permanents presents considerable difficulty, a great deal of the research effort on permanents has been directed at finding good lower and upper bounds. A first question might be to determine when an $m \times n$ ($m \leqslant n$) nonnegative matrix A has a positive permanent. It follows from a classical result of Frobenius and König that a necessary and sufficient condition is that A contain no $p \times q$ zero submatrix for any p, q with $p + q = n + 1$. A consequence of this is that a doubly stochastic matrix has a positive permanent and indeed that the set Ω_n of $n \times n$ doubly stochastic matrices is a convex polytope whose vertices are precisely the $n \times n$ permutation matrices (Birkhoff [3]). In particular, since Ω_n is compact, there exists a largest positive number c_n such that $\operatorname{per} A \geqslant c_n$ for all $A \in \Omega_n$ (the van der Waerden conjecture asserts that $c_n = n!/n^n$). There are a number of lower bounds known for the permanent of an $n \times n$ matrix of 0's and 1's. One of the more interesting is the following which was first obtained by Minc [29] using a structure theorem for matrices of 0's and 1's due to Sinkhorn and Knopp [39]. Call an $n \times n$ matrix A of 0's and 1's *fully indecomposable* if it has no $r \times s$ zero submatrix with $r + s = n$ (it follows from the Frobenius-König theorem that if A is fully indecomposable then every 1 of A is part of a nonzero term in the permanent of A). For an $n \times n$ fully indecomposable matrix A of 0's and 1's, $\operatorname{per} A \geqslant \sigma(A) - 2n + 2$ where $\sigma(A)$ is the number of 1's of A. This inequality was given a geometric interpretation and proof by Brualdi and Gibson [7]. The nonempty faces of Ω_n are in one-to-one correspondence with $n \times n$ matrices B of 0's and 1's such that for some permutation matrices P, Q, PBQ is a direct sum of fully indecomposable matrices (called a matrix with *total support*). The vertices of the face $\mathfrak{F}(B)$ corresponding to B are those permutation matrices R with $R \leqslant B$. If B is fully indecomposable, then the dimension of $\mathfrak{F}(B)$ equals $\sigma(A) - 2n + 1$. Since the number of vertices of a convex polytope of dimension k is at least $k + 1$, the above inequality follows. Moreover, equality occurs if and only if $\mathfrak{F}(B)$ is a simplex. In addition Brualdi and Gibson characterized those matrices B with total support such that $\mathfrak{F}(B)$ is a simplex.

For an $n \times n$ nonnegative matrix $A = [a_{ij}]$ with row sums $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_n$ and column sums $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n$, the inequalities

$$\operatorname{per} A \leqslant \prod_{i=1}^{n} r_i, \qquad \operatorname{per} A \leqslant \prod_{i=1}^{n} s_i$$

are immediate (since e.g. $\prod_{i=1}^{n}(\sum_{k=1}^{n} a_{ik}) = \sum_\sigma a_{1\sigma(1)} \cdots a_{n\sigma(n)}$ + other nonnegative terms). Jurkat and Ryser [21] improved these bounds by showing that

$$\operatorname{per} A \leqslant \prod_{i=1}^{n} \min\{r_i, s_i\}.$$

In 1963 Minc [28] conjectured that for A a matrix of 0's and 1's,

$$\operatorname{per} A \leqslant \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$

This conjecture was established by Brégman [4] in 1973 and a simpler proof demonstrated by Schrijver [36]. A consequence of this inequality is the validity of Ryser's conjecture that on the class of $vk \times vk$ matrices of 0's and 1's with all row and column sums equal to k the permanent function takes its maximum on the direct sum of v $k \times k$ matrices of all 1's. For $n \times n$ matrices with all row and column sums equal to k with k not a factor of n, the maximum of the permanent is not known. Foregger [12] showed that for an $n \times n$ fully indecomposable matrix A with nonnegative integral entries,

$$\operatorname{per} A \leqslant 2^{\sigma(A)-2n} + 1.$$

This implies that an $n \times n$ fully indecomposable matrix A of 0's and 1's such that the corresponding face of Ω_n has dimension k satisfies

$$\operatorname{per} A \leqslant 2^{k-1} + 1.$$

Brualdi and Gibson [7] extended this result to obtain that for any $n \times n$ matrix A of 0's and 1's with total support with the corresponding face of Ω_n having dimension k,

$$\operatorname{per} A \leqslant 2^k.$$

In both cases conditions for equality were obtained (a pyramid over a $(k-1)$-dimensional rectangular parallelepiped and a k-dimensional rectangular parallelepiped, respectively).

A fascinating approach for obtaining upper bounds for permanents of $n \times n$ complex matrices was described by Jurkat and Ryser [20]. Given an $n \times n$ matrix A they constructed $\binom{n}{i-1} \times \binom{n}{i}$ matrices $P_i(A)$, $i = 1, \ldots, n$, whose entries are constructed from the entries of row i of A such that

$$P_1(A)P_2(A) \cdots P_n(A) = [\operatorname{per} A],$$

the matrix of the right being a 1×1 matrix whose unique entry is per A. As a result for any matrix norm $\|\cdot\|$,

$$|\operatorname{per} A| \leqslant \prod_{i=1}^{n} \|P_i(A)\|.$$

Each choice of matrix norm gives rise to an upper bound for the permanent and this was exploited by Jurkat and Ryser.

An important method for obtaining inequalities for the permanent function was set forth by Marcus and Newman in their pioneering paper [27]. Let V be an n-dimensional unitary space with inner product $(\cdot, \cdot)$, and let $M_m(V)$ be the space of m-multilinear functionals on V. Let $V^{(m)}$ be the dual space of $M_m(V)$. Then $V^{(m)}$ is spanned by the decomposable tensors $f = u_1 \otimes \cdots \otimes u_m$ $(u_1, \ldots, u_m \in V)$ where for $\varphi \in M_m(V)$,

$$f(\varphi) = \varphi(u_1, \ldots, u_m).$$

An inner product is induced on $V^{(m)}$ by defining

$$(u_1 \otimes \cdots \otimes u_m, v_1 \otimes \cdots \otimes v_m) = \prod_{i=1}^{m} (u_i, v_i).$$

For $u_1, \ldots, u_m \in V$ define the *symmetric product* of $u_1, \ldots, u_m$ by

$$u_1 * \cdots * u_m = \frac{1}{m!} \sum_{\sigma} u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(m)}$$

where the summation extends over all permutations σ of $\{1, \ldots, m\}$. Now let $v_1, \ldots, v_m$ also be vectors of V and define the $m \times m$ matrix $A = [a_{ij}]$ by $a_{ij} = (u_i, v_j)$. Then straightforward calculation shows that

$$(u_1 * \cdots * u_m, v_1 * \cdots * v_m) = \frac{1}{m!} \operatorname{per} A,$$

where the inner product is that of $V^{(m)}$. This is a remarkable equation for it shows that the permanent can be represented as an inner product, and, in particular, the Cauchy-Schwartz inequality applies. From this Marcus and Newman obtained the following fundamental inequality: for $m \times n$ and $n \times m$ matrices A and B, respectively,

$$|\operatorname{per} AB|^2 \leqslant \operatorname{per} AA^* \operatorname{per} BB^*$$

and deduced from it inequalities such as $|\operatorname{per} A| \leqslant \operatorname{per} AA^*$, $|\operatorname{per} U| \leqslant 1$ for a unitary matrix U, and $\det A \leqslant \operatorname{per} A$ for a positive semidefinite hermitian matrix A (Schur's inequality). In addition they showed that the van der Waerden conjecture held for an $n \times n$ positive semidefinite hermitian doubly stochastic matrix A, that is, $\operatorname{per} A \geqslant n!/n^n$ with equality if and only if $A = J_n$. Refinements and additional inequalities using the multilinear approach were obtained by Marcus and Minc [25] and by others.

The present status of the van der Waerden conjecture is the following. The Marcus-Newman contribution to the van der Waerden conjecture mentioned above was extended by Sasser and Slater [35] to normal doubly stochastic $n \times n$ matrices the argument of whose eigenvalues λ satisfied $-\pi/2n \leqslant \arg \lambda \leqslant \pi/2n$, and further by Friedland [13] to doubly stochastic $n \times n$ matrices whose numerical range is a subset of the closed *sector* $[-\pi/2n, \pi/2n]$. For $n = 3$ the conjecture was established by Marcus and Newman [26], for $n = 3$ and 4 by Eberstein and Mudholkar [11], for $n = 5$ by Eberlein [10]. Recently Friedland [14], extending a line of investigation initiated by Bang [1] has shown that for *any* $n \times n$ doubly stochastic matrix A

$$\operatorname{per} A \geqslant e^{-n},$$

a bound virtually of the same order of magnitude as that in the van der Waerden conjecture. There also exist in the literature numerous conjectured inequalities which would imply the van der Waerden inequality. While interesting, none of these seem to be more tractable than the van der Waerden inequality itself.

According to the result of Pólya mentioned previously, for every integer $n \geqslant 3$ and every $n \times n$ matrix $M = [m_{ij}]$ of 1's and -1's, there exists an $n \times n$ matrix $A = [a_{ij}]$ such that

$$\operatorname{per} A \neq \det M * A,$$

where $M * A = [m_{ij}\, a_{ij}]$, the *Hadamard product* of M and A. This result rules out the possibility that the techniques of evaluation for determinants could be used for evaluating permanents by simply replacing some entries by their negatives. Marcus and Minc [23] generalized this considerably by showing that for $n \geqslant 3$, there is no linear transformation T on the linear space of $n \times n$ matrices such that $\operatorname{per} A = \det T(A)$ for all $n \times n$ matrices A. This negative result showed effectively that permanents could not be evaluated by means of determinants. Of course these negative findings do not rule out the possibility that certain types of matrices could be evaluated via determinants. Let $A = [a_{ij}]$ be an $n \times n$ matrix of 0's and 1's, and suppose there exists an $n \times n$ matrix $M = [m_{ij}]$ of 1's and -1's such that $\operatorname{per} A = \det M*A$. It then follows that for each permutation σ of $\{1, \ldots, n\}$ such that $a_{1\sigma(1)} \cdots a_{n\sigma(n)} = 1$, $m_{1\sigma(1)} \cdots m_{n\sigma(n)} = 1$ or -1 according as σ is an even or odd permutation. Hence for *any* $n \times n$ matrix $B = [b_{ij}]$ with the same zero pattern as A (i.e. $b_{ij} = 0$ if and only if $a_{ij} = 0$), $\operatorname{per} B = \det M*B$ so that the permanent of B can be evaluated via determinants. Thus it is of considerable interest to determine when the permanent of a matrix of 0's and 1's can be converted to a determinant by affixing a minus sign to some of its entries. Such matrices were characterized by Little [22] in the following way.

Let $X = [x_{ij}]$ be an $n \times n$ matrix of 0's and 1's such that for integers $i_1 \neq i_2$ and $j_1 \neq j_2$, $x_{i_1 j} = 1$ if and only if $j = j_1$ or j_2, $x_{ij_1} = 1$ if and only if $i = i_1$ or i_2, and $x_{i_2 j_2} = 0$. Then the $(n-1) \times (n-1)$ matrix X' obtained from X by eliminating row i_1 and column j_1 and replacing $x_{i_2 j_2}$ by 1 is said to be obtained from X by a *reduction* (a special case of *contraction* used in [7]). Since the permanent is a multilinear function of its rows, $\operatorname{per} X' = \operatorname{per} X$. A matrix Y is said to be *reducible* from a matrix X if it can be obtained by a sequence of reductions starting from X. Then the permanent of an $n \times n$ matrix A of 0's and 1's can be converted to a determinant by affixing minus signs to some of the entries of A if and only if there do not exist permutation matrices P and Q, an integer $m \leqslant n - 3$, and an $(n - m) \times (n - m)$ matrix B of 0's and 1's such that

$$I_m \oplus B \leqslant PAQ$$

where the 3×3 matrix of all 1's is reducible from B. (Such matrices B were characterized in [7] and correspond to those faces of the convex polytope of doubly stochastic matrices which are 4-dimensional 2-neighborly polytopes–the only type of 2-neighborly polytope that can occur as a face of this polytope.)

The book *Permanents* by Minc is an unlikely book, unlikely because its intent is to give an account of the function, the permanent, as it developed from its inception in 1812 to the present time. While the permanent function is an interesting, often fascinating, matrix function with applications to enumerative problems, while it has appeared in certain physical problems [8], [17], [32] and in probability theory, and while it has been used as a tool in an investigation [7] of the polytope Ω_n of $n \times n$ doubly stochastic matrices, it remains, I believe, a matrix function of specialized interest. Certainly it does not rival its relative the determinant for importance in mathematics and its applications. Besides giving a nice outline of the historical development of permanents, Minc surveys the progress attained in the last twenty years, proving many results, stating others without proof, and referring to the literature for additional ones. After reading the book I found myself disagreeing with the exclusion of certain topics and the inclusion of others. For instance, I think Brenner's result [5] that the permanent of a diagonally dominant matrix is different from zero deserves more than just an informal mention in the introductory chapter on the historical development of permanents. Since Brenner's proof is done for determinants and in some generality, Minc would have performed a valuable service by including a streamlined proof of the result for permanents. I would have preferred that the refined structure theorem for matrices of 0's and 1's derived by Hartfiel [15] be proved rather than mentioned at the end of the proof of the structure theorem of Sinkhorn and Knopp. This refined structure theorem has been very useful in obtaining inequalities for permanents and is used in the proof of Theorem 3.1 in this book. Having it would have made the proof of the inequality $\operatorname{per} A \geqslant \sigma(A) - 2n + 2$ for an $n \times n$ fully indecomposable matrix A of 0's and 1's almost trivial. Indeed since this inequality holds for nonnegative integral matrices, a shorter proof of Theorem 3.1 can be obtained. A discussion of the problem of bounds on the permanent of $n \times n$ matrices of 0's and 1's with exactly three 1's in each row and column would have been desirable. This problem has received considerable attention in the last ten years (e.g. Sinkhorn [38], Hartfiel [16]) and has resulted in methods which have proved more generally useful. Recently Voorhoeve [41] has obtained an exponential lower bound by elementary means. Since Ryser's method for evaluating permanents is superior to that referred to as the Binet-Minc method, I think the latter could have been more profitably replaced by a discussion of the use of the multilinearity of the permanent function in evaluating permanents.

The above comments should not distract from the fact that Minc has written a book that will be very valuable to researchers of the permanent. In addition, the style is such that nonspecialists will be able to peruse it to see why the permanent has interested so many people in the last twenty years. The bibliography contains 303 references and will be very useful. However, in spite of Minc's statement that it "contains every paper and book on permanents published before the end of 1977 or awaiting publication at that time", references [18] and [22] are not included in the bibliography.

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RICHARD A. BRUALDI