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Topological uniform structures, by Warren Page, Wiley, New York, 1978, xvi + 398 pp.

1. The subject. A *uniform space* is a topological space in which Cauchy's criterion for convergence makes sense; a *complete* uniform space is a uniform space in which Cauchy's criterion is true (Cauchy $\Rightarrow$ convergent). Every uniform space X may be enlarged (via a generalization of Cantor's construction of the reals from the rationals) to a complete uniform space $\hat{X}$, called its *completion*, in which it is dense. {To keep the discussion as nontechnical as possible, all topological spaces will be assumed to be Hausdorff spaces.}

The seeds of uniformity appear in *Cauchy's criterion* for the convergence of a sequence of real numbers (a_n): it is sufficient that for every $\varepsilon > 0$ there exist an index n_0 beyond which any two terms a_m, a_n of the sequence are within ε of each other. What is essential here is that there is a 'floating' notion of mutual distance $< \varepsilon$, not tied down to any particular point of the line (in particular, not tied down to a hypothetical limit of the sequence).

A second classical example of uniformity is the proposition that every continuous, real-valued function f defined on a closed interval is *uniformly continuous*: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever x and y are points of the interval within δ of each other, their images $f(x)$ and $f(y)$ under f are within ε of each other. Once again, a roving notion of mutual nearness: it does not matter, for instance, whether x and y are near the left endpoint or near the right endpoint; what matters is that they are near each other.

A third classical example is the proposition that a real power series $\sum_{k=0}^{\infty} a_k x^k$ *converges uniformly* on any closed interval $[a, b]$ interior to its interval of convergence: writing $s_n(x) = \sum_{k=0}^{n} a_k x^k$ for the partial sums of the series, the crux of the proposition is that for every $\varepsilon > 0$ there exists an index n_0 such that for $m \geqslant n_0$, $n \geqslant n_0$, one has $|s_m(x) - s_n(x)| < \varepsilon$ for all x in $[a, b]$.What is essential here is that s_m and s_n, regarded as functions on $[a, b]$, are everywhere within ε of each other; their graphs may be near to or far from the x-axis, but they are within ε of each other throughout the interval $[a, b]$. The significance of this proposition is that it opens the door for term-by-term differentiation and integration of power series, whence much of their usefulness in analysis (e.g. in the solution of differential equations).

A concept bound up so intimately with convergence, completeness and compactness is destined to permeate analysis. It does indeed permeate analysis, spilling over to create new domains of general topology having as their original objective the clarification and unification of uniformity phenomena in analysis.

The main sources of uniform structure are metric spaces, compact spaces, and topological algebraic structures. It is instructive to look first at metric spaces, which provide the most transparent examples.

Let X be a metric space, with distance function d. For each $\varepsilon > 0$, there is centered at every point of the space a ball of (uniform) radius ε. So to speak, there is in the neighborhood of all points a simultaneous notion of nearness. For each $\varepsilon > 0$ let us write $U_\varepsilon = \{(x, y) \in X \times X: d(x, y) < \varepsilon\}$. The intuitive statement "the points x and y of X are within ε of each other" has the formal meaning $(x, y) \in U_\varepsilon$; and the "open ball of radius ε centered at y" is the set $U_\varepsilon(y) = \{x: (x, y) \in U_\varepsilon\}$, consisting of all points of X that are within ε of y. To say that a sequence (x_n) in X is Cauchy, means that for every $\varepsilon > 0$ one has $(x_m, x_n) \in U_\varepsilon$ for all sufficiently large m and n; to say that the sequence converges to a point x means that for every $\varepsilon > 0$, one has $(x_n, x) \in U_\varepsilon$ for all sufficiently large n. Thus, questions of convergence and completeness may be expressed in terms of the family $(U_\varepsilon)_{\varepsilon>0}$ of sets of ordered pairs of points of the space X, i.e., a family of subsets of the cartesian product $X \times X$.

The idea of the theory of uniform structures is to start with a set X and a specified set $\mathcal{U}$ of subsets of $X \times X$. The set $\mathcal{U}$ is required to satisfy certain axioms. The axioms are natural, few in number, and lead quickly to a topology on X; they reflect, in a set-theoretic way, the familiar properties of a metric space (notably symmetry and the triangle inequality). We need not set down the list of axioms explicitly; instead, the sets of $\mathcal{U}$ will be described fully for the various examples to be discussed. {The best place to learn about uniform structures is in Chapter II of Bourbaki's *Topologie générale* [**1**], my nominee for the best-written mathematics textbook of all time.} The elements of $\mathcal{U}$ are called the *entourages* for the uniform structure, and the set $\mathcal{U}$ of all entourages is called the *uniformity* of the uniform structure. When (X, d) is a metric space, the entourages for its uniform structure are the subsets of $X \times X$ that contain some U_ε; that is, $U \in \mathcal{U}$ means that there exists an $\varepsilon > 0$ such that $U_\varepsilon \subset U \subset X \times X$.

A recurrent trait of uniform structures is that one often seems to get more than one bargained for–something extra, free of charge. Here is one of many examples. A topological space is said to be *uniformizable* if its topology arises from a uniform structure. A priori, all we know is that the space is Hausdorff (T_2); in fact, every uniformizable space must be *completely regular* $(T_{3^1/2})$, and, conversely, every completely regular space is uniformizable [**1**, Chapter IX, §1, $n°$ 5, Theorem 1]. This is quite startling: the uniformizable spaces (defined by an innocent list of set-theoretical axioms) are, so to speak, the topological spaces whose topology is determined by the continuous *real-valued* functions that they admit.

Every compact space X is uniformizable, with unique uniformity $\mathcal{U}$, consisting of all neighborhoods of the diagonal $\{(x, x): x \in X\}$ in the product topological space $X \times X$. It is elementary that every subspace of a uniformizable space is uniformizable; in particular, every subspace of a compact space is uniformizable. On the other hand, every completely regular space is a subspace of a compact space (Stone-Čech compactification). Thus, *a topological space is uniformizable if and only if it is a subspace of a compact space* [**1**, Chapter IX, §1, $n°$ 5, Proposition 3]. In particular (one-point compactification) every locally compact space is uniformizable; it is perhaps surprising, and certainly important, that the uniformity of a locally compact space need not be unique (more about this later).

We have seen how uniformity arises in metric spaces and in compact spaces; let us now see how it arises in topological algebraic structures. The most important topological algebraic structures are the topological groups. (Even in topological vector spaces, rings, modules, algebras and fields, questions of uniformity refer to the underlying additive topological group structure.) A *topological group* is a group G (say with multiplicative notation) equipped with a topology for which the mappings $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous mappings $G \times G \to G$ and $G \to G$, respectively. For each element a of G, the mapping $x \mapsto ax$ (called left-translation by a) is a homeomorphism of G; thus if V runs over the set of neighborhoods of the neutral element e of G, then aV runs over the set of neighborhoods of the point a. On the other hand, for fixed V and variable a, the family $(aV)_{a \in G}$ is a family of neighborhoods of "uniform size V" of the points of G. A uniformity $\mathfrak{U}_l$ on G is obtained by letting the V's play the role of ε's: for each neighborhood V of e, one defines $U_V = \{(x, y) \in G \times G\colon x^{-1}y \in V\}$; $\mathfrak{U}_l$ consists of the sets $U \subset G \times G$ such that $U \supset U_V$ for some neighborhood V of e. The uniformity $\mathfrak{U}_l$ is said to define the *left uniform structure* of G. From the formula $aV = \{y \in G\colon (a, y) \in U_V\}$ one infers that the topology on G derived from the uniformity $\mathfrak{U}_l$ coincides with the original topology on G. (Consideration of right-translations $x \mapsto xa$ and neighborhoods Va leads to the *right uniform structure* of G, with uniformity denoted $\mathfrak{U}_r$.) In particular, every topological group G is a uniformizable topological space. A priori, G is only assumed to be a Hausdorff space, but the compatibility of its topology with the group operations has produced an unexpected bonus: *every topological group is completely regular*. Here is another example of a topological bonus. Recall that every metrizable space is first-countable (that is, each point has a denumerable fundamental system of neighborhoods). For topological groups, the converse is true (Birkhoff-Kakutani theorem): *every first-countable topological group is metrizable*. (This is really a theorem about uniform structures having a denumerable fundamental system of entourages [**1**, Chapter IX, §2, *n*° 4 Theorem 1].)

Uniformity makes many striking appearances in analysis. Here are some memorable examples, drawn mostly from the theory of locally compact groups.

Every locally compact topological group is completely regular ($T_{3 1/2}$), for two reasons: because it is locally compact, and because it is a topological group. In fact, *every locally compact group is paracompact*, hence normal (T_4) [**1**, Chapter III, §4, *n*° 6, Proposition 13 and Chapter IX, §4, *n*°4, Proposition 4].

The left and right uniformities of a topological group G need not coincide; when they do, let us say that G is *bi-uniform*. {Examples: G compact (uniqueness of uniformity!), or discrete, or abelian.} Let $\hat{G}$ be the completion of G with respect to its (say) left uniform structure. In general, the operations of G cannot be extended so as to make $\hat{G}$ a topological group, but all is well when G is bi-uniform (this sufficient condition is not necessary). For locally compact groups, there is no problem at all: *they are already complete* [**1**, Chapter III, §3, *n*° 3,Corollary 1 of Proposition 4].

Every locally compact group possesses a left-invariant measure (and a right-invariant measure), unique up to proportionality, called the left (resp. right) *Haar measure* of the group. Uniformity properties figure so essentially in the proof that there is a generalization to group actions on uniformly locally compact spaces (theorem of I. E. Segal **[4]**).

A sparkling example of the role of uniformity at the crossroads of algebra and analysis is the following beautiful theorem of R. Godement. Let G be a locally compact group, with left Haar measure μ, and let $L^2(G)$ be the Hilbert space of complex-valued functions on G square-integrable with respect to μ. There is associated with G an algebra $\mathcal{Q}$ of operators in $L^2(G)$, analogous to the group algebra of a finite group (called the left von Neumann algebra of G). Suppose, in addition, that μ is also a right Haar measure (groups with this property are called unimodular). Godement's theorem: In order that every left-invertible element of the ring $\mathcal{Q}$ be right-invertible, it is necessary and sufficient that G be bi-uniform ($\mathcal{U}_l = \mathcal{U}_r$), **[3**, p. 46, Theorem 6].

One of the cornerstones of functional analysis is the Baire category theorem: *Every complete metric space is a Baire space* (i.e., every nonempty open subset is of the second category). This leads to the Banach-Steinhaus "uniform boundedness principle", and to Banach's "open-mapping theorem" and "closed graph theorem". At the base of the pyramid: uniformity.

As a concluding example of uniformity at work, I cite the following pretty result of Robert Ellis. Let G be a group equipped with a locally compact topology for which multiplication is separately continuous (i.e., for each $a \in G$, the translation mappings $x \mapsto ax$ and $x \mapsto xa$ are continuous). Ellis' theorem **[2]**: *G is a topological group*. The proof is a deft exploitation of the topology of uniform convergence on compact sets, and the fact that every locally compact space is a Baire space.

The concept of uniform structure is due to André Weil; all of the results on uniformizability cited above already appear in his 1938 monograph **[5]**. {Weil's monograph serves as a blueprint for Chapter II of Bourbaki's (then yet to be published) *Topologie générale*, as well as for parts of Chapters III and IX.}

2. The book. The emphasis of the book under review is revealed in the chapter headings: Uniform spaces (66 pp.), Topological groups (65 pp.), Topological vector spaces (120 pp.), Topological algebras (66 pp.), Abstract harmonic analysis (45 pp.). The emphasis is not on general uniform structures, to which only the first one-sixth of the book is devoted; nor is the emphasis on uniformity considerations that occur in various applications; the emphasis is on topological algebraic structures, with (despite the page-count of the third chapter) the center of gravity in abstract harmonic analysis (duality theory of locally compact abelian groups).

In the opening chapter, several concepts more general than uniformity are exercised for the sake of perspective (quasi-uniformity, local uniformity, semi-uniformity), but the chapter quickly settles down to developing the fundamentals of the theory of uniform structures. The second chapter includes a detailed construction of Haar measure in locally compact groups; there is an interesting discussion of characters of arbitrary locally compact groups (a technical prelude, mostly topological, to the final chapter). In the third chapter, besides the standard topics on topological vector spaces (Hahn-Banach, Krein-Milman, duality), there is a special emphasis on topics pertaining to completeness (hence to uniformity). The fourth chapter includes standard material on normed algebras and Banach algebras, with and without involution, as well as not-so-standard material on more esoteric topological algebras (locally m-convex Q-algebras, etc.). The fifth and final chapter culminates in a proof of the Pontrjagin duality theorem. The prerequisites for reading the book (drawn mainly from general topology and integration theory) are sketched in three brief appendices.

Proofs are detailed and carefully done. The layout is excellent; the printer deserves a medal for his skill in representing the many, often intricate and unusual, notations. The text is heavy on special symbols and terminology; since these are usually defined once and used from then on without explanation, the burden on the reader's power of concentration builds quickly. (The burden on the proofreader's concentration was more than occasionally overwhelming.) An index of symbols and a good general index help, but the reader's task is still formidable (the browser's, hopeless).

The author states in his Preface: "This work is reasonably self-contained and accessible to students with a background in elementary analysis, linear algebra and point set topology. At the same time it covers a good amount of advanced material without going off into the purple deep." The reviewer concurs; there is a lot of fine material in this book for second-year graduate courses and seminars.

Every mathematician needs to speak a little topology. The message of this book is that every analyst needs to speak a little uniformity; it is a central language of analysis, not just a peripheral dialect. Uniform structures deserve a niche in every first-year graduate course in general topology; this book effectively demonstrates why.

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General theory of Lie algebras, by Yutze Chow, Gordon and Breach, New York, 1978, Volume 1, xxii + 461 pp., Volume 2, xx + 436 pp., $72.00.

1. Among the three main types of nonassociative algebras, Lie, alternative and Jordan algebras, the Lie algebras were the first to be studied and are still the most important because of their connections with other parts of mathe-