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Mathematical logic. An introduction to model theory, by A. H. Lightstone, Mathematical Concepts and Methods in Science and Engineering, Vol. 9, Plenum Press, New York and London, 1978, xiii + 338 pp.,$22.50.

1. In the past twenty years or so mathematical logic has moved from being a subject often considered rather exotic (if indeed it was really mathematics) to being a subject about which most mathematicians ought to know at least a little. The reasons are not hard to find. First, mathematical logic essentially enshrines the idea of precision in mathematical language. Second, it treats of the logical processes of deduction and makes clearer the abstract structure of arguments. Third, the techniques involved lead to new developments in and of other parts of mathematics.

Precision of language was encouraged, even demanded, by the nineteenth century crises in analysis and, later, set theory. (How easy it is now to distinguish between convergence: $\forall\varepsilon > 0\ \forall x\ \exists\delta > 0 \ldots$ and uniform convergence $\forall\varepsilon > 0\ \exists\delta > 0\ \forall x \ldots$ How tricky for Cauchy.)

The analysis of deduction culminates in the provision of a neat (essentially finite) presentation of axioms and rules which give only true statements and, in certain cases, all true statements (the completeness theorems).

New techniques emerged including the ideas of recursive functions and the development of computer programs. (These together with the precision of language led to an unexpected answer to e.g. Hilbert's tenth problem: there is no general formal technique which will decide diophantine problems.) Recursive function theory comes from the ideas of formal languages; the other aspect, truth, leads to model theory: the semantic aspect of the languages. Most present general interest here centres on nonstandard analysis. Abraham Robinson's brilliantly simple observation was to apply a reasonably well-known theorem (compactness) in what appeared an entirely unpromising situation.

2. Corresponding to the three aspects of logic noted above (though not in one-one correspondence) are three theorems.

Propositional calculus deals only with logical connectives (e.g. and, or, not) applied to unanalyzed statements and is very useful as a pedagogical prelude. The first theorem (completeness of propositional calculus) shows that a finite set of axioms (or schemes for axioms) and rules will yield all and only those (formal) combinations of statements which are true independent of the truth of the constituent statements.

Predicate calculus goes further allowing analysis of statements into subject and predicate (and thereby allowing talk of functions) and also admitting the quantifiers $\forall$ and $\exists$ (now quite familiar in informal notation). Again a completeness theorem holds paralleling that above for propositional calculus. However, Henkin's proof leads to a different, and more used, formulation. A structure $\mathfrak{A}$ (think of an algebraic one such as a group for concreteness) is a *model* for a collection Σ of sentences if every sentence of Σ is true in $\mathfrak{A}$. (The axioms of group theory are almost completely formal in present-day notation and any group is a model of those axioms.) The second theorem, the Gödel-Henkin completeness theorem, says: A set Σ of sentences is consistent if, and only if, it has a model.

Because of the finite nature of the way predicate calculus is presented the third theorem, the *compactness theorem*, follows almost at once:

If every finite subset of a collection Σ of sentences has a model, so does Σ.

Robinson's move was, in essence, to take all the true sentences Σ about, say, the real numbers and add a new symbol **c**, say, with axioms $0 < \mathbf{c} < 1/n$ for every positive integer n. Now (any finite subset of) Σ with any finite number of the extra axioms has a model (use the reals just as before and interpret **c** as any number less than the nonzero minimum of the *finite* number of fractions $1/n$ mentioned in the finite number of new axioms). We can therefore apply the compactness theorem to get a model of the theory of the reals which nevertheless has infinitesimals. From this, nonstandard analysis can be steadily developed replacing compactness arguments of standard analysis with (to most of us) more perspicuous ones using infinitesimals.

3. The book under review covers these three theorems, devotes another forty pages to nonstandard analysis and another sixty to other topics: a total of 333 pages of text. It is explicitly presented as an introduction to model theory and nonstandard analysis and thus goes a little further than Lightstone's earlier book (**1964**). It has many competitors ranging from adequate to excellent (see the references).

Let me recall the three points with which I began. 1. Logic enshrines precision of language. 2. It clarifies the structure of arguments. 3. It provides useful new techniques.

The following give some idea of Lightstone's book for those who might consider it for use as a basic text in logic.

"For a more rigorous [sic] proof of this [the completeness] theorem [for propositional calculus], see Henkin (**1949**)" (p. 102).

On p. 221 ff. he only proves the completeness theorem for predicate calculus for countable languages leaving it as an exercise (Ex. 1, p. 224) with only p. 102, see above, as a guide. But the completeness theorem for *uncountable* languages is precisely what is required for his application to nonstandard analysis in the succeeding chapter. Certainly it can be argued that in an elementary text such a move is justifiable, but it contrasts strangely with the slow treatment of everything else.

Now motivation My students always find logic exciting and perplexing (though I, personally, usually do not give them their first logic course!). When Mendelson's now classic text was published, Goodstein wrote in Mathematical Reviews (**29** #2158) "Motivation [is] adequate"–damning by faint praise. The present book contains virtually nothing to link it with the rest of mathematics–or logic–for a couple of hundred pages. That is asking a lot of the reader unfamiliar with logic.

For the reader who is a logician a note of warning. Many of the definitions are highly nonstandard. Thus the definition of elementary extension on p. 124, though technically equivalent to the (standard) one in, say, Chang-Keisler (1973), might cause some confusion as Lightstone allows constants in his formulae for *all* elements of the models.

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Topos theory, by Peter Johnstone, London Mathematical Society Monographs, vol. 10, Academic Press, London, New York, San Francisco, 1977, xxiii + 367 pp., $34.25.

At last there is at hand in this book a systematic and complete (though austere) presentation of the recent extensive developments in category theory leading to the notion of an elementary topos. This notion arose by the confluence of two separate trends, from geometry and from logic.

On the one hand, Grothendieck had observed that a topological space X can be studied in terms of its sheaves F. Indeed, he replaced X by the category $\mathrm{Sh}(X)$ of all sheaves F of sets on X, and called this category a *topos*, on the grounds that *this* was what the topologists need.

To define a sheaf F on a topological space X, one does not need the *points* of the space, but only its *open sets* U and their coverings by other open sets.