with the slow treatment of everything else.

Now motivation My students always find logic exciting and perplexing (though I, personally, usually do not give them their first logic course!). When Mendelson's now classic text was published, Goodstein wrote in Mathematical Reviews (29 #2158) "Motivation [is] adequate"–damning by faint praise. The present book contains virtually nothing to link it with the rest of mathematics–or logic–for a couple of hundred pages. That is asking a lot of the reader unfamiliar with logic.

For the reader who is a logician a note of warning. Many of the definitions are highly nonstandard. Thus the definition of elementary extension on p. 124, though technically equivalent to the (standard) one in, say, Chang-Keisler (1973), might cause some confusion as Lightstone allows constants in his formulae for *all* elements of the models.

REFERENCES

C.-C. Chang and H. J. Keisler (1973), *Model theory*, Amsterdam, New York.

L. A. Henkin (1949), *The completeness of the first-order functional calculus*, J. Symbolic Logic **14**, 159–166.

A. H. Lightstone (1964), *The axiomatic method*, Englewood Cliffs, N. J.

Some elementary texts:

First the classic:

E. Mendelson (1964), *Introduction to Mathematical logic*, Princeton Univ. Press, Princeton, New Jersey.

Others include:

J. L. Bell and A. Slomson (1969), *Models and ultraproducts: An introduction*, North-Holland, Amsterdam-London.

H. B. Enderton (1972), *A mathematical introduction to logic*, Academic Press, New York.

A. G. Hamilton (1978), *Logic for mathematicians*, Cambridge Univ. Press, London and New York.

D. Hilbert and W. Ackermann (1950), *Mathematical logic*, Chelsea, New York.

P. S. Novikov (1964), *Elements of mathematical logic*, Oliver and Boyd, Edinburgh-London; Addison-Wesley, Reading, Mass.

J. N. Crossley

Topos theory, by Peter Johnstone, London Mathematical Society Monographs, vol. 10, Academic Press, London, New York, San Francisco, 1977, xxiii + 367 pp., $34.25.

At last there is at hand in this book a systematic and complete (though austere) presentation of the recent extensive developments in category theory leading to the notion of an elementary topos. This notion arose by the confluence of two separate trends, from geometry and from logic.

On the one hand, Grothendieck had observed that a topological space X can be studied in terms of its sheaves F. Indeed, he replaced X by the category $Sh(X)$ of all sheaves F of sets on X, and called this category a *topos*, on the grounds that *this* was what the topologists need.

To define a sheaf F on a topological space X, one does not need the *points* of the space, but only its *open sets* U and their coverings by other open sets.

The typical example is the sheaf C which assigns to each open set U the set $C(U)$ of all continuous real-valued functions on U. In general, a "presheaf" F assigns to each open U a set $F(U)$ and to each inclusion $V \subset U$ a "restriction" map $F(U) \to F(V)$, written $f \mapsto f|V$ as if it were the restriction of a function f–and with the corresponding properties for composition of restrictions. Such a presheaf is a sheaf if, for every covering $\{U_i | i \in I\}$ of an open set U, each $f \in F(U)$ can be put together *uniquely* from its "pieces" $f_i = f|U_i$–provided only that on each intersection $U_i \cap U_j$ the piece f_i matches the piece f_j. The sheaves, defined thus in terms of coverings of open sets, suffice to get the cohomology of X (Chapter 8). For example, in the category of all sheaves of sets one can also construct a sheaf A of abelian groups; it is simply a sheaf of sets with suitable operations $+ : A \times A \to A$ and $- : A \to A$, satisfying the appropriate identities, as expressed by commutative diagrams.

Thus the topos $\text{Sh}(X)$ suffices for cohomology of a topological space. For cohomology groups in algebraic geometry one needs the more general "Grothendieck topologies". Here the open sets of X are replaced by the objects of any category C, and the inclusions $V \subset U$ by arrows $V \to U$ of C. With the appropriate properties for "coverings", one has made C into a site with coverings J and one has a definition of sheaf so as to produce again a category $\text{Sh}(C, J)$. Such a category is still called a "topos", sometimes a "Grothendieck topos".

Pursuing this idea, Giraud found necessary and sufficient conditions that a given category be such a topos, for some "site" (C, J).

On the other hand, in 1964, Lawvere had raised a question of foundations: Could one replace the usual Zermelo-Fraenkel axioms on *membership* in sets by suitable axioms on functions between sets, that is, axioms on the *category* **Set** of sets? His initial axiom system of this type was cumbersome. Then in 1969–1970 Lawvere and Tierney at Dalhousie University observed that Giraud's conditions for categories of sheaves, when moderately altered and weakened, could be stated in *elementary* terms, using only first order logic. These weakened axioms describe what is now called an "elementary" topos. It can be the category of sheaves of sets on some topological space or on a site for some Grothendieck topology; in particular, it can be the ordinary category of all sets (= sheaves on a one-point space). It can also be the category of all presheaves on a space, where a presheaf is just any functor on the open sets of the space, with no matching conditions on coverings. Still more generally, it can be the category of all functors to sets from *any* small category C (not just some category C of open sets). A typical example is the category of all diagrams $X_0 \to X_1 \to X_2 \to \cdots$ of sets X_i; such a diagram can be considered as a set X "varying through time" 0, 1, 2, 3, . . . just as a sheaf of sets can be viewed as a set varying through space. In other words, one can axiomatize the category of sets effectively by choosing first axioms which *also* apply to other similar categories such as sheaf and functor categories. This idea frees set theory from its prior (and stiff) uniqueness.

Once recognized, this line of ideas had a rapid and sometimes confusing development, hard for an outsider to follow except by diligent study of successive issues of lecture notes. Happily, all this development, up to 1977, is now captured and codified in Johnstone's monograph. It begins with a

hard-hitting historical introduction: "I do not . . . like Grothendieck . . . view topos theory as a machine for the demolition of unsolved problems in algebraic geometry . . . but I do believe that the spreading of the topos-theo-retic outlook into many areas of mathematical activity will inevitably lead to the deeper understanding of the real features of a problem which is an essential prelude to its correct solution".

The reader may need a deeper study of the book to see what this means, but the understanding is there. On an elementary level, there is for example the observation of Lawvere that a J-indexed family of sets X_j for $j \in J$ is best viewed as a single set X (the disjoint union $\coprod X_j$) with a map $X \to J$. Technically, with J fixed, this leads to the recognition that the category of all functions $f: X \to J$, with arrows the evident maps over J between two such functions, is an elementary topos; moreover, this applies for f, X, and J in any given topos $\mathcal{E}$. In other words, the original function $J \to \mathbf{Set}$ given by $j \mapsto X_j$ is replaced by a function $X \to J$. Similarly, a construction due to Grothendieck replaces a functor $F: C \to \mathbf{Set}$ on a (small) category C by a category D mapped onto C by a projection functor $P: D \to C$; explicitly, D is the category of all the "elements" $\{\langle x, c \rangle, x \in F(c)\}$ of the functor F, while the functor $P\langle x, c \rangle = c$ satisfies a special condition (is a "discrete opfibra-tion"). This observation is much more than a simple translation of one description $F: C \to \mathbf{Set}$ of a set-valued functor into another equivalent description. For, if we replace the category of sets in this context by a more general topos $\mathcal{E}$, we can still describe what we mean by a small category $\mathbf{C}$ "internal" to $\mathcal{E}$. The objects of $\mathbf{C}$ are represented by one object C_0 of $\mathcal{E}$, the *object-of-objects*, while the arrows (or morphisms) of $\mathbf{C}$ are represented by C_1, the objects-of-arrows. This description is supplemented by two arrows $C_1 \to C_0$ giving the domain and codomain, plus an arrow for composition, satisfy-ing the usual associative law, expressed in diagrammatic form. However, this concept of an internal category does not allow *any* corresponding way of describing, by diagrams or otherwise, a functor F mapping the internal category C into the (large) external category $\mathcal{E}$. However, there is an *alternative* description of such a functor F by *another* internal category D and a suitable "internal diagram" $P: D \to C$. With this description, one readily can form the external category $\mathcal{E}^{\mathbf{C}}$ of all such internal diagrams. In other words, for any internal category $\mathbf{C}$ in a topos $\mathcal{E}$, one has a category $\mathcal{E}^{\mathbf{C}}$ whose objects are all the internal functors P on $\mathbf{C}$ to $\mathcal{E}$. This is like the usual "functor categories" of all set-valued functors on a category $\mathbf{C}$; these functor categories are known to be useful in many ways, especially in homological algebra. Moreover, we have (Chapter 2) the theorem that $\mathcal{E}$ a topos makes $\mathcal{E}^{\mathbf{C}}$ a topos, just as in the case of sets.

This straightforward, better understanding of set-valued functors is in its turn a tool to the better understanding of the theorem of Giraud, characteriz-ing those categories which are categories of sheaves on some topological space or on some Grothendieck topology. This is done (see below) by way of a "relative" Giraud theorem which is more general ("sets" replaced by any topos) and at the same time simpler.

The topos-theoretic approach also provides such better understanding in many other connections: In the construction from a presheaf of its associated

sheaf; in the very meaning of a Grothendieck topology as used in the definition of a sheaf; in the presence in each presheaf category of the "algebra" of its open sets, a Heyting algebra (= intuitionistic variant of Boolean algebra); in the consequent presence of an "internal" intuitionistic logic (the Mitchell-Benabou language) to express definitions and properties in any topos; in the observation that the double negation operator $\neg\,\neg$ in this internal logic is at the same time a *topology* on the topos; and in the discovery that sheafification for *this* topology is the heart of Paul Cohen's proof of the independence of the continuum hypothesis–all items to be discussed below.

The axioms for a topos depend on a similar better understanding of the "universal" properties of the basic constructions of set-theory. For example, the set 1 with just one element can be described as a terminal object in the following sense: For any set X, there is exactly one function $X \to 1$. Originally, the "pullback" of a function $g\colon Y \to Z$ along another function $f\colon X \to Z$ is described as the set P of those pairs $\langle x, y \rangle$ of elements with $fx = gy$; instead it now appears as the vertex P in a square diagram

$$
\begin{array}{ccc}
P & \to & Y \\
\downarrow & & \downarrow f \\
X & \underset{g}{\to} & Z
\end{array}
$$

which is "universal" for given f and g in the sense that for any different choice P' of P, there will be a unique $P' \to P$. Such pullbacks include products (take $Z = 1$). If a category has all pullbacks and a terminal object, then it has all finite (projective) limits.

There is a similar better understanding of the meaning of the characteristic function f of a subset S of a set A. As usual, one defines f by setting $f(x) = 0$ or 1 according as $x \in S$ or $x \notin S$. Then $f\colon X \to \{0, 1\}$ maps X to the set $\Omega = \{0, 1\}$ of "truth values". The inclusion $t\colon \{0\} \to \{0, 1\}$ is a "typical" subset; indeed a universal one, because the definition of the characteristic function f of S says precisely that S is the *pullback* of t along f, as in the diagram

$$
\begin{array}{ccc}
S & \to & 1 \\
m\downarrow & & \downarrow t \\
A & \underset{f}{\to} & \Omega
\end{array}
$$

where $1 = \{0\}$ is the "terminal" object in **Set**. The corresponding axiom for an elementary topos $\mathcal{E}$ now states that each such topos $\mathcal{E}$ has an object Ω called the *subobject classifier*, and an arrow $t\colon 1 \to \Omega$, called "true", such that every monomorphism $m\colon S \to A$ in $\mathcal{E}$ is the pullback of true along a unique arrow $f\colon A \to \Omega$. One may think of f as the "property" of A whose "extension" is the subobject S. With this formulation, functor categories and sheaf categories, etc., all have subobject classifiers. For example, in the category of sheaves on a topological space X, the subobject classifier Ω is the sheaf which assigns to each open U in X the set of all open subsets of X, while Ω' for presheaves has $\Omega'(U)$ the set of all sieves (of open sets) on U.

Johnstone's monograph is more systematic. The axioms on a topos $\mathcal{E}$ require that $\mathcal{E}$ be a category with all finite limits, with for each X an

exponential (a functor) $(\ )^X$ right adjoint to $(\ ) \times X$: $\mathcal{E} \to \mathcal{E}$, and with a subobject classifier, as above. These axioms *are* elementary, when the limits and adjoints are described (as with Ω above) by their universal properties. In short order, Johnstone exhibits the power of this combination of axioms, proving first a variety of elementary facts: That equivalence relations R are "effective", i.e. that quotient sets of equivalence classes exist; that the singleton map $X \to \Omega^X$ is monic; and that partial maps, i.e. maps defined on a subset of X to Y are representable; that is, can be obtained by pullback from a suitable monic $Y \rightarrowtail \tilde{Y}$. Since subobjects of A correspond to arrows $A \implies 1$ $\times A \to \Omega$ and thence, by the definition of the exponent, to arrows $1 \to \Omega^A$, the assignment $A \mapsto \Omega^A$ is the contravariant power-set functor P. This leads up to Paré's observation that P is right adjoint to its opposite functor P^{op}, hence that P is monadic (triplable), and hence by the crude version of Beck's triplability theorem, that $\mathcal{E}^{\mathrm{op}}$ is a category of algebras over $\mathcal{E}$.

Since $\mathcal{E}$ has all finite limits, so does the category $\mathcal{E}^{\mathrm{op}}$ of P-algebras. Therefore the topos $\mathcal{E}$ has all finite colimits. Thus one has the result, first proved by a direct method by Mikkelsen, that every elementary topos has not only finite limits, but also finite colimits (initial object 0, coproducts, and "pushouts"). Previously this had been assumed as one of the axioms for an elementary topos, while a Grothendieck topos was required to have coequalizers of equivalence relations and (infinite) coproducts.

The subobject classifier Ω functions as an "object of truth values". For sets, $\Omega = \{0, 1\}$, with the usual two truth values, as noted above. For other toposes Ω is *not* usually two-valued. For sheaves, $\Omega(U)$ is the set of open subsets of U; since the work of Stone in the 1930's, this has been recognized as a Heyting algebra ($=$ a Brouwerian lattice) (note in particular that the complement $\neg V$ of V open in U need not satisfy $\neg \neg V = V$!). In a general topos Ω is an object and not a set, but it still is always a Heyting-algebra object, in that one can define operations such as

$$\bigwedge : \Omega \times \Omega \to \Omega, \quad \neg : \Omega \to \Omega$$

and $\bigvee$, $\implies$ which satisfy the diagrammatic versions of the axioms for a Heyting algebra. Johnstone's presentation disperses these definitions (§1.49, 3.1, and Chapter 5) and so may hide the natural topological origin of the algebra, but the definitions themselves are simple: For example, intersection $\bigwedge$ is the characteristic function of the subobject $t \times t$: $1 \times 1 \to \Omega \times \Omega$; false: $1 \to \Omega$ is the characteristic function of the subobject $0 \to 1$, and $\neg$ is the characteristic function of "false". With this start, Johnstone develops in §5.4 the explicit Mitchell-Benabou language for a topos $\mathcal{E}$, which includes quantifiers and which allows formulation of internal (intuitionistic) properties in any topos, and which satisfies all the usual axioms and rules of inference of intuitionistic predicate logic, save *modus ponens* (p. 155). This clearly represents a connection with logic, worthy of deeper understanding.

Johnstone's Chapter 3 starts with a quick (and almost deadpan) introduction of the beautiful notion of topology in a topos. First consider the topos of presheaves on a space X; there, as already noted, the subobject classifier Ω is the functor with $\Omega(U)$ the set of all sieves on U (a sieve being a collection of open subsets V of U, containing with each V all open subsets of V).

Grothendieck's observation about coverings draws attention to the set $J(U)$ of all those sieves on U which *cover* U. Thus $J \subset \Omega$ is a subfunctor (a subpresheaf) and so has a characteristic funtion j: $\Omega \to \Omega$. Moreover, j has three characterizing properties:

$$j^2 = j, \ j \text{ true} = \text{true}, \ j \bigwedge = \bigwedge (j \times j): \Omega \times \Omega \to \Omega.$$

These three properties (or the corresponding commutative diagrams) capture exactly the Grothendieck-Artin definition of a "Grothendieck topology". Hence a *topology* on a topos $\mathcal{E}$ is arrow j: $\Omega \to \Omega$ with these three properties. Far though they may seem from the ordinary notion of a topological space, it is a simple and perspicuous definition. Johnstone shows in short order how this definition and the standard procedures lead to a definition of those objects in $\mathcal{E}$ which are sheaves relative to the topology j and to the proof that these objects constitute a topos $\mathrm{Sh}_j(\mathcal{E})$ with subobject classifier the image of j. The proof that the inclusion function $\mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$ has a left adjoint, "sheafification", is more subtle; Johnstone gives the proof from his thesis, which is ingeniously adapted from the Grothendieck proof. The chapter ends with an illuminating list of examples; in particular, in any topos $\mathcal{E}$ the map $\neg \neg$: $\Omega \to \Omega$ given by double negation is a topology. Thus the intuitionist Brouwer meets the topologist Brouwer!

A *geometric morphism* f: $\mathcal{E} \to \mathcal{E}'$ of toposes (Chapter 4) is a pair of functors f_*: $\mathcal{E} \to \mathcal{E}'$ and f^*: $\mathcal{E}' \to \mathcal{E}$ such that f^* is left adjoint to f_* and left exact as well. The name arises from topology, since a continuous map f: $X \to Y$ of spaces carries each sheaf G on X, regarded as a functor on open sets, forward to a sheaf $f_* G$ on Y and also carries each sheaf G' on Y, regarded as an "espace étale" over Y, backward by pullback to a sheaf $f^* G'$ on Y. Johnstone notes many other examples of geometric morphisms; in particular the inclusion $\mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$. (Logical morphisms of toposes are functors preserving all the structure; they play a lesser role.) If $\mathcal{E}$ is a fixed topos, one is led naturally to the category $\mathbf{Top}/\mathcal{E}$ of "toposes over $\mathcal{E}$"; its objects are geometric morphisms f: $\mathcal{F} \to \mathcal{E}$ and its arrows g: $\mathcal{F} \to \mathcal{F}'$ are geometric morphisms with $f'g = f$. For example, if $\mathbf{C}$ is an internal category in $\mathcal{E}$, the category $\mathcal{E}^{\mathbf{C}}$, already noted above, is a topos over $\mathcal{E}$. Diaconescu's theorem (1975) plays a central role; given f: $\mathcal{F} \to \mathcal{E}$, it shows that the geometric morphisms g: $\mathcal{F} \to \mathcal{E}^{\mathbf{C}}$ over $\mathcal{E}$ correspond exactly to those internal presheaves G on $f^*\mathbf{C}$ which are flat, in the sense that G regarded as a category is filtered. Johnstone gives a most perspicuous proof of this theorem: Given g, the associated flat presheaf is constructed as the pullback $g^* Y$ of the Yoneda bifunctor Y; on the other hand, given G one constructs the geometric morphism g by describing its "inverse image" half g^*: $\mathcal{E}^{\mathbf{C}} \to F$. Explicitly, g^* sends each internal diagram D in $\mathcal{E}^{\mathbf{C}}$ to $f^*D \otimes G$. Here $\otimes$ is Benabou's elegant tensor product of functors.

In the topos of all sets, the arrows x: $1 \to X$ correspond exactly to the elements of X. Moreover, these elements suffice to distinguish functions. Specifically, if $f \neq g$: $X \to Y$, there is an element x: $1 \to X$ with $fx \neq gx$; we say that the object 1 *generates* the topos **Set**. This simple situation need not obtain in a more general topos: Each object X still has "elements" x: $1 \to X$, but 1 need not generate the topos. A set T of objects of $\mathcal{E}$ is said to *generate*

the topos $\mathcal{E}$ when to any two different arrows $f \neq g$: $X \rightarrow Y$ of $\mathcal{E}$ there exists an object G of T and an arrow h: $G \rightarrow X$ with $fh \neq gh$. For example, in the topos $\mathrm{Sh}(X)$ of sheaves F on a topological space X, the terminal object 1 is the constant sheaf 1, so an "element" $1 \rightarrow F$ is just a global cross-section of the sheaf F. Two different maps f,g: $F \rightarrow F'$ need not differ on a global cross-section, so 1 does not generate. However, the subsheaves of the sheaf 1 are essentially just the open sets U of the space X, and $f \neq g$ means that there is some open set $U \subset X$ with $f_U \neq g_U$ and hence a map h: $U \rightarrow F$ with $fh \neq gh$. Hence in $\mathrm{Sh}(X)$ the subobjects of 1 form a set of generators. This illustrates the way in which a general topos diverges from the topos of sets.

Giraud's theorem, as already suggested, characterizes those categories $\mathcal{E}$ which are equivalent to the category $\mathrm{Sh}(C, J)$ of sheaves on the site (C, J) of some Grothendieck topology; the Grothendieck toposes $\mathcal{E}$ so characterized are always elementary toposes, but not conversely. In the preliminary comments to his book, Johnstone states this Giraud characterization (four exactness conditions, plus two smallness conditions; small hom sets and a *set* of generators); he also sketches, rather cryptically, the Giraud proof. The main text then illuminates this whole situation by using the Diaconescu theorem to give Diaconescu's proof of a simple and more general "relative Giraud theorem". First note that the global cross-section functor Γ: $\mathrm{Sh}(C, J) \rightarrow \mathbf{Set}$ is (the direct part) of a geometric morphism; moreover, up to a canonical isomorphism, this is the only geometric morphism from $\mathrm{Sh}(C, J)$ to $\mathbf{Set}$.

The Giraud theorem then reduces to the statement that an elementary topos $\mathcal{E}$ is a Grothendieck topos if and only if it has a geometric morphism to the topos $\mathbf{Set}$ and also has an object G whose subobjects generate $\mathcal{E}$. This notably simplifies the Giraud theorem. The "relative" version now replaces $\mathbf{Set}$ by an arbitrary topos $\mathcal{E}$. It states that a geometric morphism f: $\mathcal{F} \rightarrow \mathcal{E}$ is naturally isomorphic to the composite $\mathcal{F} = \mathrm{Sh}_j(\mathcal{E}^C) \rightarrow \mathcal{E}^C \rightarrow \mathcal{E}$ for some internal category C on $\mathcal{E}$ and some topology j if and only if f has an "object of generators" G–one with the property that to each X in $\mathcal{F}$ there is an object Y in $\mathcal{E}$, a subobject S of $f^*Y \times G$ and an epimorphism $S \twoheadrightarrow X$ (when $\mathcal{E} = \mathbf{Set}$, $f^*Y \times G$ is just a Y-indexed copower of G in F). The proof is long but illuminating.

In 1964 Lawvere observed that the Peano axioms on the 0: $1 \rightarrow N$ and successor functions s: $N \rightarrow N$ for the set N of natural numbers could be replaced by the simple statement that $1 \xrightarrow{0} N \xrightarrow{s} N$ is universal among diagrams $1 \xrightarrow{a} X \xrightarrow{u} X$ in $\mathbf{Sets}$, meaning that there is a unique g: $N \rightarrow X$ with $g(0) = a$ and $gsn = ugn$. The very form of this description provides at once for the definition by recursion of functions like g–much more effectively than in the usual constructions from the axioms of Peano. The same Lawvere axiom describes a "natural number object" N in any topos and so provides, as in set theory, an axiom of infinity for a topos. Given such an N one constructs, in a straightforward way, addition and multiplication in N, finite cardinals, the object $\mathbf{Q}$ of natural numbers and the object $\mathbf{R}$ of real numbers–with the Cauchy reals a *sub*object (p. 215) of the Dedekind reals! There are more surprising uses of the natural number object N; for example, its presence in $\mathcal{E}$ is equivalent (Theorem 6.41) to the existence of free monoids in $\mathcal{E}$, and is the starting point for universal algebra in $\mathcal{E}$. Moreover,

following Wraith, one can construct from $\mathcal{E}$ with N a new topos $\mathcal{E}[U]$ with a designated object U and a geometric morphism $\mathcal{E}[U] \to \mathcal{E}$ so that U is an *object classifier*: any object V in any topos $\mathcal{F}$ over $\mathcal{E}$ can be had (up to equivalence) by pulling U back along a unique geometric morphism $\mathcal{F} \to \mathcal{E}[U]$ over $\mathcal{E}$. Intuitively, $\mathcal{E}[U]$ is the free topos obtained by adjoining to $\mathcal{E}$ an indeterminate object U. This is an analogy to the 1972 construction by Hakim of a topos over $\mathcal{E}$ containing a ring object R which was similarly a *ring*-classifier, universal for commutative ring objects in toposes over $\mathcal{E}$. Here a commutative ring may be regarded as a model (in a topos) of the usual finitary *algebraic* theory of rings. But, as Johnstone observes, we frequently have to consider objects with a structure which is not purely algebraic in this sense, in that the structure is defined by formulas which are not simply equational, as for example in the case of local rings. Now a local ring A can be described as a commutative ring with zero 0, identity element e and a "group of units" $U \subset A$ such that $0 \to 1$ is the equalizer of the two maps 0, e: $1 \to A$ and such that for all a in A, either a or $e - a$ is in U. The logical formulas expressing these two properties are not algebraic equations, but have the happy property that they are preserved not just by logical morphisms of toposes, but also by the inverse image maps of geometric morphisms of toposes. This leads to the precise definition (p. 199) of a *finitary geometric theory* stated by formulas (using $\wedge$, $\vee$, and $\exists$, but not $\neg$, $\Rightarrow$, or $\forall$) and thus to the proof of the decisive theorem (Joyal, Benabou, Tierney) on classifying topoi: If T is a finitely presented finitary geometric theory and $\mathcal{E}$ a topos with a natural number object N, there exists a topos $\mathcal{E}[T]$ over $\mathcal{E}$, which is a classifying topos for T, in the sense that there is an equivalence between models of T in a topos $\mathcal{F}$ over $\mathcal{E}$ and the category of geometric morphisms of $\mathcal{F}$ into $\mathcal{E}[T]$ over $\mathcal{E}$. From this theorem, Johnstone goes on to discuss the work of Hakim and J. C. Cole on spectra for a topos, and to indicate some of the fascinating further questions in these directions.

These developments emphasize the connections from topos theory to algebraic geometry and universal algebra. From the original Lawvere axiomatic viewpoint, there are also close relations with set theory. Clearly the axioms for an elementary topos have more models than (and hence are much weaker than) the Zermelo-Fraenkel axioms for set theory. So, following Freyd, define a *well-pointed topos* $\mathcal{E}$ to be one in which the terminal object 1 is (as in sets) a generator while the unique map $0 \to 1$ is not an isomorphism. This compares directly with "weak Zermelo set theory" as described by E. J. Thiele; the axioms are extensionality, empty set, unordered and ordered pairs, the power set, union sets, foundation (i.e. regularity), and the *restricted* comprehension axiom (the usual comprehension, but only for formulas having quantifiers which are "restricted", as in $(\forall s) \, s \in x$ or $(\exists t) \, t \in y$). Such a restricted comprehension does include the usual uses of the comprehension axioms in mathematics. Then if M is a model of weak Zermelo set theory, the category $\mathcal{S}(M)$ of sets and functions in M is a well-pointed topos.

The converse construction requires some way of constructing sets (with their elements!) within a topos–where there do not appear to be elements. This issue is met by using sets T which are transitive, in the sense that $x \in y \in T$ implies $x \in T$. From any set S one may construct, via the

replacement axiom of Zermelo-Fraenkel, the set T of all members of members of S, thus embedding S in a transitive set T. The membership relation R restricted to a transitive set T is both extensional and well founded, and these two properties, recast in terms of the possibility of a suitable recursion, suffice to define in any topos a "transitive" object T–an object X with an extensional and "well-founded" binary relation r. Now from a topos $\mathcal{E}$ one constructs "sets" as suitable equivalence classes of tuples (X, r, m), where X with r is a transitive object and $m: 1 \to \Omega^X$ is a "global element" of the "power set" of X. These sets $S(\mathcal{E})$ provide a model of weak Zermelo set theory. Together $M \mapsto S(M)$ and $\mathcal{E} \mapsto S(\mathcal{E})$ compare set theory and topos theory. To get the best comparison, add to weak Zermelo set theory two easy consequences of replacement (every set can be embedded in a transitive set and any set with an extensional well-founded binary relation is isomorphic (Mostowski, 1949) to a transitive set. Similarly we add to the axioms of a well-pointed topos the requirement (judiciously formulated; see p. 314) that every object of $\mathcal{E}$ can be embedded in a transitive object. Then we get the definitive theorem, due to Cole, Mitchell, and Osius, that $S(SM)$ is isomorphic to M and $S(S\mathcal{E})$ is equivalent to $\mathcal{E}$. This provies equiconsistency between well-pointed topos theory and weak Zermelo set theory.

Other corresponding axioms can be added to both sides of this comparison. For instance, the axiom of choice for a topos $\mathcal{E}$ is the statement that every epimorphism in $\mathcal{E}$ splits (has a left inverse). This axiom implies that that topos is Boolean (Theorem 5.23). This theorem of logic was first proved, by Diaconescu, in its topos-theoretic form.

The Cohen proof of the independence of the continuum hypothesis provides another striking connection with logic (Chapter 9). One starts with a model **Set** of set theory, with N, its power set Ω^N, and a still larger set I which is to be "forced" to fall within Ω^N. Thus each $i \in I$ is to be a subset of N. One starts with a finite set P of conditions to this, of the form $m \in i$ or $n \notin j$ for $m, n \in N$ and $i, j \in I$. The set P of all these p is a poset, hence a (small) category. From it one constructs the functor category **Set**^{P}. In this topos, as in every topos, $j = \neg\,\neg$ is a topology. Forming the sheaves for this topology yields another topos $\mathcal{F} = Sh_{\neg\,\neg}(\textbf{Set}^P)$ which is Boolean, i.e. the Heyting algebra object Ω is a Boolean algebra, because $\neg\,\neg$ is now the identity, but not yet two-valued. Much as in the classical case, one then finds a suitable ultrafilter and uses the "calculus of fractions" to divide $\mathcal{F}$ out by this ultrafilter so as to construct from $\mathcal{F}$ a two-valued Boolean topos in which the continuum hypothesis suitably fails.

This brief outline of this elegant proof applies *mutatis mutandis* both to the original Cohen proof and to the Lawvere-Tierney topos-theoretic variant. The use of double negation appears explicitly in the Cohen papers, but only in retrospect does one see the presence of sheaves for the double negation topology. The topos-theoretic version needs suitable auxiliary techniques, most critically the construction (5.46) within the exponent object Y^X (the "set" of all functions $X \to Y$) of an object Epi(X, Y) of all epimorphisms $X - Y$. At any rate, the analysis of forcing in a critical case raises the question as to how and whether numerous other examples of set-theoretic forcing might be reduced and understood better via the construction of sheaf

toposes in place of Boolean-valued models. So far this has been done (by M. Bunge; see p. 329) only for the independence of the Souslin hypothesis.

These are by no means the only connections with logic. Deligne's theorem, that every "coherent" topos has enough "points" (Theorem 7.44), is intimately related to the Gödel-Henkin completeness theorem for finitary first order theories, and there is (Theorem 7.16) a similar categorical version of the Lowenheim-Skolem theorem. In other words, topos theory not only developed from a collision of algebraic geometry and set theory, but this collision has set off various other surprises: Sheaves appearing in set theory and completeness theorems in algebraic geometry. Other connections–with cohomology theory, with torsors, and with profinite fundamental groups–are left for the reader to discover in Johnstone's book.

This book does provide good examples of the better understanding promised in the introduction. To achieve this understanding, the reader must on occasion study hard, to get at what is behind the economical presentation, with little motivation, of all the techniques and corresponding theorems. Only by choosing this austere presentation was the author able to bring all these (and many other ideas) in the brief compass of 360 pages.

There is a very helpful index of notation at the back. Given the range of theorems collected from many authors reported here, usually in neater and quicker ways, I located very few slips; Theorem 0.14 from Eilenberg and Moore is misquoted, while Theorem 7.37(i) from Grothendieck on coherent topoi is misproved; both can be corrected by reference to the original sources. Lemma 9.17 is misnumbered–but enough of such carping comments. This is a dense and rich book, which has organized valuable material as an aid to our deeper understanding of sheaf theory, logic, and algebra.

SAUNDERS MAC LANE

Symmetry and separation of variables, by Willard Miller, Jr., Addison-Wesley Publishing Company, Reading, Massachusetts, 1977, xxx + 285 pp., $21.50.

Separation of variables is a technique for solving special partial differential equations. It is taught in elementary courses on partial differential equations, but the method usually does not achieve the status of a mathematical theory.

Because most references do not give a precise definition of separation of variables, I invented a definition myself. Let us call a partial differential equation in n variables $x_1, \ldots, x_n$ *separable* if there are n ordinary differential equations in $x_1, \ldots, x_n$, respectively, jointly depending on $n - 1$ independent parameters (the separation constants), such that, for each choice of the parameters and for each set of solutions $(X_1, \ldots, X_n)$ of the o.d.e.'s, the function $u(x_1, \ldots, x_n) := X_1(x_1) \cdots X_n(x_n)$ is a solution of the p.d.e. Under the terms of this definition a converse implication often holds: If $u = X_1 \cdots X_n$ is a factorized solution of the p.d.e. then, for some choice of the parameters, the X_i's are solutions of the o.d.e.'s. The most familiar cases of separability deal with a linear second order p.d.e. which separates into n