toposes in place of Boolean-valued models. So far this has been done (by M. Bunge; see p. 329) only for the independence of the Souslin hypothesis.

These are by no means the only connections with logic. Deligne's theorem, that every "coherent" topos has enough "points" (Theorem 7.44), is intimately related to the Gödel-Henkin completeness theorem for finitary first order theories, and there is (Theorem 7.16) a similar categorical version of the Lowenheim-Skolem theorem. In other words, topos theory not only developed from a collision of algebraic geometry and set theory, but this collision has set off various other surprises: Sheaves appearing in set theory and completeness theorems in algebraic geometry. Other connections–with cohomology theory, with torsors, and with profinite fundamental groups–are left for the reader to discover in Johnstone's book.

This book does provide good examples of the better understanding promised in the introduction. To achieve this understanding, the reader must on occasion study hard, to get at what is behind the economical presentation, with little motivation, of all the techniques and corresponding theorems. Only by choosing this austere presentation was the author able to bring all these (and many other ideas) in the brief compass of 360 pages.

There is a very helpful index of notation at the back. Given the range of theorems collected from many authors reported here, usually in neater and quicker ways, I located very few slips; Theorem 0.14 from Eilenberg and Moore is misquoted, while Theorem 7.37(i) from Grothendieck on coherent topoi is misproved; both can be corrected by reference to the original sources. Lemma 9.17 is misnumbered–but enough of such carping comments. This is a dense and rich book, which has organized valuable material as an aid to our deeper understanding of sheaf theory, logic, and algebra.

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Symmetry and separation of variables, by Willard Miller, Jr., Addison-Wesley Publishing Company, Reading, Massachusetts, 1977, xxx + 285 pp., $21.50.

Separation of variables is a technique for solving special partial differential equations. It is taught in elementary courses on partial differential equations, but the method usually does not achieve the status of a mathematical theory.

Because most references do not give a precise definition of separation of variables, I invented a definition myself. Let us call a partial differential equation in n variables $x_1, \ldots, x_n$ *separable* if there are n ordinary differential equations in $x_1, \ldots, x_n$, respectively, jointly depending on $n - 1$ independent parameters (the separation constants), such that, for each choice of the parameters and for each set of solutions $(X_1, \ldots, X_n)$ of the o.d.e.'s, the function $u(x_1, \ldots, x_n) := X_1(x_1) \cdots X_n(x_n)$ is a solution of the p.d.e. Under the terms of this definition a converse implication often holds: If $u = X_1 \cdots X_n$ is a factorized solution of the p.d.e. then, for some choice of the parameters, the X_i's are solutions of the o.d.e.'s. The most familiar cases of separability deal with a linear second order p.d.e. which separates into n

linear o.d.e.'s of second or first order. Usually one can express the general solution of a linear separable p.d.e. as a sum or integral of factorized solutions. Suitable boundary conditions may also be taken into account. Other variants of separation of variables can be obtained by changing the functional dependence of u on the X_i's, for instance $u(x_1, \ldots, x_n) = X_1(x_1) + \cdots + X_n(x_n)$ or $u = R^{-1}X_1 \cdots X_n$ for some nonvanishing function R of $x_1, \ldots, x_n$. In the latter case we have the so-called R-separability.

The two-variable Helmholtz equation in polar coordinates,

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} + \omega^2 u = 0,$$

provides a simple example of separation of variables. Let $u(r, \theta) = f(r)g(\theta)$. Then u satisfies the equation above if and only if, for some value of k, f and g satisfy the o.d.e.'s

$$r^2 f''(r) + rf'(r) + (\omega^2 r^2 - k^2)f(r) = 0, \qquad g''(\theta) + k^2 g(\theta) = 0,$$

respectively. The solutions of the first o.d.e. are linear combinations of the Bessel functions $J_k(\omega r)$ and $J_{-k}(\omega r)$. This example illustrates the way many special functions arose in history: as factorized solutions of the p.d.e.'s of mathematical physics when written in separable coordinates.

In a systematic study of separation of variables one can ask two natural questions. First, find necessary and sufficient conditions for a p.d.e. to be separable. Second, if some specific p.d.e. is given, classify all transformations of the independent variables such that the p.d.e. becomes separable.

A criterium for separability was first obtained by Stäckel **[11]**, at the end of the nineteenth century. Curiously enough, he did it for a nonlinear equation occurring in classical mechanics, the Hamilton-Jacobi equation. Consider an n-dimensional Riemannian manifold with orthogonal local coordinates $x_1, \ldots, x_n$. Then the corresponding fundamental tensor (g_{ij}) is diagonal. In case of constant potential energy, the associated Hamilton-Jacobi equation is

$$\sum_{i=1}^{n} (g_{ii})^{-1}\left(\frac{\partial u}{\partial x_i}\right)^2 = \lambda, \tag{1}$$

where λ is a positive constant. According to Stäckel's criterium, equaton (1) is separable by solutions of the form $u = X_1(x_1) + \cdots + X_n(x_n)$ iff there is a matrix-valued function $\Phi(x_1, \ldots, x_n) := (\Phi_{ij}(x_i))$ ($n \times n$ matrix) such that $g_{ii}^{-1} = \lambda(\Phi^{-1})_{1i}$, $i = 1, \ldots, n$.

Robertson **[10]** in 1928 considered a quantum mechanical analogue to Stäckel's problem, namely separability for the time-independent Schrödinger equation. Consider a Riemannian manifold M with (g_{ij}) diagonal as above, $g := \prod_{i=1}^{n} g_{ii}$, and Δ the Laplace-Beltrami operator. For constant potential energy Robertson's problem reduces to the question of separability for the Helmholtz equation

$$\Delta u + \lambda u := \sum_{i=1}^{n} g^{-1/2}\frac{\partial}{\partial x_i}\,\frac{g^{1/2}}{g_{ii}}\,\frac{\partial u}{\partial x_i} + \lambda u = 0 \tag{2}$$

on M. It follows from Robertson's paper that (2) is separable by solutions of the form $u := X_1(x_1) \cdots X_n(x_n)$ iff (i) the Stäckel condition holds, i.e., $g_{ii}^{-1} = \lambda(\Phi^{-1})_{1i}$ for some matrix $\Phi := (\Phi_{ij}(x_i))$, and (ii) $g^{1/2}/\det \Phi = f_1(x_1) \cdots f_n(x_n)$ for certain functions f_i. Next Eisenhart **[2]** observed that, if (i) holds, then condition (ii) is equivalent to the vanishing of the Ricci tensor

R_{ij} off the diagonal. Thus for Einstein spaces and, in particular, for spaces of constant curvature, (2) is separable iff the Stäckel condition holds. Similar results can be derived for pseudo-Riemannian manifolds.

All literature dealing with these separability criteria fails to give a precise definition of separation of variables. As a consequence, at some stage of the usual proofs of a necessary condition for separability one has some "evident" corollary of the separability which is not evident at all, but rather acts as a precise substitute for the original vague definition of separation of variables. Examples of this in the book under review can be found on p. 14, formula (2.25) and p. 15, case II, first sentence. The only precise definition of separation of variables I found in the literature is in a paper by Niessen [**8**, p. 329]. However, his definition already specifies the particular form which a separable p.d.e. must have, a property which one would prefer to obtain as a corollary of the definition. In a forthcoming paper I will propose a definition (an elaboration of the second paragraph of this review) which satisfies the three criteria of being precise, being close to the informal notion of separation of variables and leading to the necessary and sufficient conditions obtained by Robertson.

In the classification of all separable coordinate systems for some specific p.d.e. the Stäckel condition is an important tool. Let me mention a few equations for which the classification has been rendered. The two-variable Helmholtz equation has four essentially different orthogonal separable coordinate systems: Cartesian, polar, parabolic and elliptic, cf. §1.2 of Miller's book. For the three-variable Helmholtz equation there are eleven such coordinate systems, a result first proved by Eisenhart [**2**]. In all cases the coordinate surfaces are quadrics, possibly degenerate. R-separable coordinates which are not separable do not occur for this equation. However, the Laplace equation in three variables admits six additional R-separable coordinate systems, for which the coordinate surfaces are cyclides, certain algebraic surfaces of fourth degree. Bôcher [**1**] in 1894 already obtained these systems, although he did not exhibit a rigorous classification of all R-separable coordinates. Standard references for these matters are Morse and Feshbach [**7**, §5.1] and Moon and Spencer [**6**].

Olevski [**9**] classified the separable coordinate systems for the Helmholtz equation on a three-dimensional space of constant positive or negative curvature. In recent times mathematicians have made progress towards a similar classification for equations on four-dimensional spaces. For instance, Kalnins and Miller [**4**] find 368 conformally inequivalent orthogonal coordinates for which the wave equation $u_{tt} - \Delta_3 u = 0$ admits an R-separation of variables. Recently there is also increased interest in the classification of nonorthogonal separable coordinate systems: see Havas [**3**] for a historical survey.

Let me next discuss the group-theoretic approach to separation of variables, which is the main theme of Miller's book. This approach starts with the observation that, if the Helmholtz equation (2) is separable, there are n linearly independent, commuting partial differential operators $S_1 = \Delta$, $S_2, \ldots, S_n$ such that their joint eigenfunctions with eigenvalues $\alpha_1 = -\lambda$, $\alpha_2, \ldots, \alpha_n$ are precisely the factorized solutions of (2). Here $\alpha_2, \ldots, \alpha_n$ can be interpreted as the separation constants. Hence, the class of all separable coordinate systems for (2) can be brought into correspondence with a subclass of the class of all $(n-1)$-tuplets of commuting second order partial differential operators which commute with Δ. Now assume that the group G of isometries of the Riemannian manifold M under consideration is a Lie group. Its Lie algebra $\mathfrak{g}$ consists of first order differential operators on M which commute with Δ. Let $L_1, \ldots, L_p$ be a basis for $\mathfrak{g}$. Let $\mathbb{S}$ be the linear space which is spanned by the second order differential operators $L_iL_j + L_jL_i$, $i, j = 1, \ldots, p$. The space $\mathbb{S}$ is included in the space of all second order operators commuting with Δ. Let us make the assumption, valid for most examples considered in Miller's book, that both spaces coincide. Let $\tilde{\mathbb{S}}$ be the class of all n-dimensional subspaces of $\mathbb{S}$ which consist of mutually commuting operators and which contain Δ. G acts on $\tilde{\mathbb{S}}$, so $\tilde{\mathbb{S}}$ can be partitioned into G-orbits. Identify coordinate systems on M which can be obtained from each other by isometries and by transformations of the form $(x_1, \ldots, x_n) \rightarrow (x_1'(x_1), \ldots, x_n'(x_n))$. We conclude that the set of essentially different separable orthogonal coordinate systems on M is in one-to-one correspondence with a subset of the set of G-orbits on $\tilde{\mathbb{S}}$. For the two- and three-variable Helmholtz equations this correspondence is very nice: each G-orbit on $\tilde{\mathbb{S}}$ corresponds to a separable coordinate system. However, for the two-variable Klein-Gordon equation $u_{tt} - u_{xx} + \omega^2 u = 0$ there is one G-orbit on $\tilde{\mathbb{S}}$ which fails to have this property, cf. p. 55 in Miller's book. It is not yet understood why such orbits occur. (However, see [13].)

Among the separable coordinate systems one distinguishes between "good guys" and "bad guys". The good guys are the subgroup coordinates. They correspond to commuting second order differential operators which are either squares of elements in $\mathfrak{g}$, i.e. related to one-parameter subgroups, or Casimir operators for nonabelian connected closed subgroups of G. The bad guys are all other separable coordinates and they usually represent the generic case. Still, a subgroup characterization of all separable coordinates is often possible if one also considers nonconnected subgroups, which may arise as the symmetry groups of the separable coordinate system, cf. Miller, Patera and Winternitz [5]. This suggests a third type of classification which is relevant for finding separable coordinate systems: find all closed subgroups of a given Lie group G.

In applications it is often important to know explicitly the kernel $c(\alpha, \beta)$ which connects two families $\{f_\alpha\}$ and $\{g_\beta\}$ of factorized solutions of (2) corresponding to two different separable coordinate systems. The connection formula is $f_\alpha = \Sigma_\beta$ (or $\int_\beta$) $c(\alpha, \beta) g_\beta$. Sometimes this problem can be elegantly handled in a Hilbert space context. Let $\tilde{S} = \{\Delta, S_2, \ldots, S_n\}$ and $\tilde{T} = \{\Delta, T_2, \ldots, T_n\}$ be the two commuting families in $\mathbb{S}$ which have the f_α's respectively the g_β's as eigenfunctions. Suppose that some subspace $\mathcal{H}$ of the solution space of (2) has a Hilbert space structure such that G acts on $\mathcal{H}$ as an irreducible unitary representation π. Then $\tilde{S}$ and $\tilde{T}$ act on $\mathcal{H}$ as families of symmetric, usually unbounded operators. Suppose these operators are selfadjoint. We restrict our attention to those f_α's and g_β's which are (possibly generalized) eigenvectors for $\tilde{S}$ respectively $\tilde{T}$ acting on $\mathcal{H}$. Next we transfer the problem of expanding f_α as a sum or integral of g_β's to some other Hilbert space $\mathcal{H}'$ on which G acts by an irreducible unitary representation π' equivalent to π and where it may be easier to deal with the corresponding families $\{f'_\alpha\}$ and $\{g'_\beta\}$. For instance, in the case of the two-variable Helmholtz equation (cf. §1.3 of Miller's book) the problem is transferred to the Hilbert space $\mathcal{H}'$ of L^2-functions on the unit circle, where π' is some induced representation. The families $\{f'_\alpha\}$ then consist of fairly simple special functions or distributions on the unit circle.

The method described above for finding the overlaps between families of factorized solutions is one of the important merits of the group-theoretic approach to separation of variables. We mention two possible applications of this method which are not considered in the book. First, the kernels $c(\alpha, \beta)$ connecting two families $\{f_\alpha\}$ and $\{g_\beta\}$ of factorized solutions are a source of special orthogonal systems, discrete or continuous. Thus new orthogonal systems of special functions or new interpretations of known systems may arise. Second, if one knows the action of suitable elements from $\mathfrak{g}$ and $\mathcal{S}$ on the families $\{f_\alpha\}$ and $\{g_\beta\}$, one may derive difference or differential or other functional equations for the kernel $c(\alpha, \beta)$ connecting these families.

The Hilbert space method of finding connection formulas cannot always be applied. For instance, it fails in the case of the Laplace equation. Some alternative methods are presented in the book, in particular Weisner's method, which involves series expansions of analytic functions and local rather than global group actions. I did not yet mention separation of variables for parabolic equations. A long chapter in Miller's book deals with two- and three-variable Schrödinger and heat equations. The Hilbert space method applied to the two-variable Schrödinger equation $iu_t + u_{xx} = 0$ is very interesting. The metaplectic representation of the semidirect product of a covering group of $SL(2, \mathbf{R})$ with the three-dimensional Heisenberg group occurs in this context.

Winternitz and Fris [12] initiated the group-theoretic approach to separation of variables. During the past five years Miller, Kalnins and collaborators wrote an impressive series of papers on this topic. In the preface to his book Miller describes the method as "a group-theoretic machine that, when applied to a given differential equation of mathematical physics, describes in a rational manner the possible coordinate systems in which the equation admits solutions via separation of variables and the various expansion theorems relating the separable (special function) solutions in distinct coordinate systems". One can handle this machine as soon as one has seen it operating for a few special equations and this is the way the book proceeds. Successively it treats the two-variable Helmholtz and Klein-Gordon equation, the Schrödinger and heat equations, the three-variable Helmholtz, Laplace and wave equation, and the Lauricella function F_D. The first example, the two-variable Helmholtz equation, is very well chosen. All the main ingredients of the method are already present here. Some further examples illustrate certain complications which may arise in applying the method and, in these examples, the author also provides some alternative techniques.

The book is aimed at a general audience rather than at specialists. According to the general editor Gian-Carlo Rota this is part of the philosophy of the Encyclopedia of Mathematics and its Applications. By general audience one has to understand all users of mathematics. The present volume is typically an applied mathematician's book, both by the lack of rigor and by the absence of general theorems. The lack of mathematical rigor is not a serious defect because, at suitable locations, it is indicated how one can formulate precise proofs. However, the very content of this book forcibly suggests the necessity for a deductive rather than inductive theory of separation of variables, starting with theorems of wide applicability and then going down to the special equations. I hope that this book will arouse the interest of some pure mathematicians, and that an interaction between the pure and applied point of view will lead to such a theory.

In my opinion, the relationship between separation of variables and group theory could have been more fully exploited in the book. For instance, for each new equation the laborious classifications of separable coordinate systems and of G-orbits on $\tilde{\mathbb{S}}$, are performed (or referred to) independently. Only after this effort it is observed that both classifications are related. The a priori knowledge that the factorized solutions of (2) must be eigenfunctions of $n - 1$ commuting operators which commute with Δ is never used. As a final point of criticism, the book lacks a solid historical foundation. The important Stäckel criterium is not even mentioned. In spite of these criticisms I like this book very much as a pioneering work in a promising field.

Miller's book is the first in the section of the Encyclopedia dealing with those special functions which occur in the practice and applications of mathematics. The section editor Richard Askey wrote a most readable foreword describing the numerous interactions between special functions and other fields from work done in previous centuries up to still unpublished results.

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