Herglotz–let's call him R. C.–ran up against a complicated definite integral in 5 dimensions. With great effort he reduced it to a much simpler integral in 3 dimensions, and this he passed on to Herglotz for his assistance in computing. After a while Herglotz came back with the comment that if by a nonobvious substitution one transforms the integral into a 5-dimensional one–which was R. C.'s original expression–then the computation is trivial. And he proceeded to show him how trivial.

On the personal side, Herglotz had great charm and was a perfect gentleman. The volume cites testimonials for that, and I can add the following corroboration. From what I can remember, I never had epistolary or personal encounters with Herglotz except for meeting him once, in May 1932, when I was in Göttingen for a lecture. (It was on Greek mathematics, and I own a clipping from a leading Berlin newspaper reporting on it.) My return train to Munich departed at 2 a.m. By academic seniority Herglotz towered over me, but he came to the station to see me off. And he stayed with me until the train started moving.

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Choice sequences, A chapter of intuitionistic mathematics, by A. S. Troelstra, Clarendon Press, Oxford, 1977, ix + 170 pp., $10.95.

The book under review is based on lecture notes of a course on choice sequences given by the author at Oxford in 1975. Choice sequences are a paradigm case of specifically intuitionistic notions, that is notions which cannot be classically understood, and one of the principal virtues of the book is that it demonstrates the possibility of coherent reasoning about such notions.

The simplest sort of "freely" chosen sequence is a lawless sequence (of natural numbers). Such sequences are necessarily incomplete, only finite initial segments having been constructed at any time. At future times one is completely free in the choice of additional elements. If Γ is any well-defined operation on such sequences η whose values are completed objects x (so we have a proof that $\forall\eta\exists!x\Gamma(\eta) = x$) then Γ is continuous in the product topology; for a proof that $\Gamma(\eta) = x$ can depend only on a finite initial segment of η. Since the axiom of choice is, intuitionistically, logically valid, we have for well-defined relations R the principle of $\forall\alpha\exists x$-continuity:

$\forall\alpha\exists x\, R(\alpha, x) \to \exists$ continuous $\Gamma\, \forall\alpha\, R(\alpha, \Gamma(\alpha))$ for α ranging over lawless sequences. By similar reasoning, we have for well-defined properties P the principle of open data: $\forall\alpha(P(\alpha) \to \exists$ open $\mathcal{O} \ni \alpha\forall\beta \in \mathcal{O}P(\beta))$.

Lawlike sequences provide a contrast with lawless ones. Lawlike sequences are given by a rule or law and in this sense are completed objects. However, operations Γ on such sequences may depend on the presentation of their arguments and consequently may not be extensional (i.e., may fail to satisfy $\forall n\, a(n) = b(n) \to \Gamma(a) = \Gamma(b)$). A fortiori they may fail to be continuous. For example, consider the paradigm case of a lawlike sequence, namely, recursive functions. There is a recursive 1-1 mapping of Gödel numbers of recursive functions into the natural numbers but there is no extensional one. Lawlike sequences enjoy one useful property not shared by lawless ones, namely, closure under lawlike continuous operations (by open data, even $2 \cdot \eta$ is never lawless).

If we require closure under even the most trivial lawlike continuous operations then there will be other sorts of choice sequences also. For example, $2 \cdot \eta$ is not lawlike either.

Collections of choice sequences closed under specified continuous operations and satisfying certain continuity and choice principles are the setting for intuitionistic analysis. Such "species" are not easy to construct. In particular, it is difficult to find nontrivial sets of continuous operations which suffice for analysis and do not conflict with continuity principles. For example, if $\mathfrak{U}$ is the closure of all (n-tuples of) lawless sequences under all lawlike continuous operations then $\mathfrak{U}$ does not satisfy $\forall\alpha\exists x$-continuity (4.13, p. 65). Even for the species of all choice sequences $\forall\alpha\exists x$-continuity has not been proved rigorously.

The author pays a great deal of attention to this problem in Chapter 4 and appendices B and C. In particular, in appendix C he presents a notion of choice sequence satisfying $\forall\alpha\exists x$-continuity and the principle of analytic data: $\forall\alpha(P(\alpha) \to \exists$ continuous $\Gamma\exists\beta\alpha = \Gamma(\beta) \wedge \forall\gamma P(\Gamma(\gamma)))$. The author has recently informed the reviewer that he and his student G. F. van den Hoeven have constructed from lawless sequences and lawlike operations a species of choice sequences closed under a nontrivial set of continuous operations and satisfying a strong form of continuity ($\forall\alpha\exists\beta$) for a large class of relations.

One striking aspect of intuitionism is its subjectivist viewpoint. The author prefers to stress that intuitionism treats concepts which do not lend themselves to classical mathematics, and that it supplements, rather than conflicts with, other approaches. For example, lawless sequences certainly do not lend themselves to the classical approach since, by $\forall\alpha\exists x$-continuity, $\neg\forall\eta\exists x(\eta(x) = 0 \vee \neg\exists x\, \eta(x) = 0)$. The author includes a very interesting, although tentative, chapter (Chapter 6) on the mathematical uses of choice sequences. The idea is that the same theorem can sometimes be proved both nonconstructively and by means of (classically inconsistent) continuity axioms, and that sometimes the use of continuity axioms can be (mechanically) eliminated from the second proof yielding a classically valid constructive one.

More precisely, CS, the theory of choice sequences built around $\forall\alpha\, \exists x$-continuity and analytic data (p. 77) and IDB, the classically valid intuitionistic theory of inductively defined continuous ("Brouwer") operations (p. 31), satisfy the following "elimination" theorem:

There is a recursively defined "elimination" map σ from formulae of CS to formulae of IDB satisfying

(1) $\sigma(A) = A$ for formulae A of IDB,

(2) $\mathrm{CS} \vdash A \leftrightarrow \sigma(A)$ and

(3) $\mathrm{CS} \vdash A \Leftrightarrow \mathrm{IDB} \vdash \sigma(A)$

(pp. 80–83).

The author applies this method to a number of well-known results including the Lindelöf and Heine-Borel covering theorems, sequential continuity implies continuity, and the Riemann permutation theorem. What is not included, except for a brief discussion (p. 99), is any sort of detailed comparison of this method of constructionization with other more familiar ones such as functional interpretations (p. 99). A hopeful, if only bibliographical, sign for the author's method is that his principal applications do not seem to appear in Bishop's book [**1**]. It might also be hoped that continuity axioms could be used to simplify, e.g., shorten proofs. However it appears that such a shortening could not be dramatic, e.g. from n to $\log n$, since elimination appears to increase length only by a quadratic amount (σ increases logical complexity by only a linear amount). In short, the role of continuity axioms in constructivizing proofs is very much an open question.

The book also contains a chapter (Chapter 7) on the completeness problem for intuitionistic predicate logic. Since the book was published some of the main results of this chapter have been considerably improved by H. Friedman [**3**], [**4**] (also [**2**, Theorem 13, p. 288]).

In addition, deSwart (1974) has appeared as [**7**] and the reference to Kreisel (1967) on p. 102 should be to [**5**, especially the top of p. 319]. The remaining chapters are: 1. Preliminaries, 2. Lawless Sequences, 3. Metamathematics of Lawless Sequences and 5. Choice Sequences. There are also a number of appendices including a short history of choice sequences.

To sum up, the book under review raises a number of interesting questions about choice sequences and demonstrates the possibility of coherent reasoning about them. It is well written and rich in detail. (There are a number of typographical errors in the book but most do not cause difficulty. The similarity of the italic 'a' and 'α' is disconcerting.)

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