MATHEMATICS OF COMPUTATION
Volume 66, Number 218, April 1997, Pages 547–572
S 0025-5718(97)00838-7

A PRIORI ERROR ESTIMATES FOR NUMERICAL METHODS FOR SCALAR CONSERVATION LAWS. PART II: FLUX-SPLITTING MONOTONE SCHEMES ON IRREGULAR CARTESIAN GRIDS

BERNARDO COCKBURN AND PIERRE-ALAIN GREMAUD

ABSTRACT. This paper is the second of a series in which a general theory of *a priori* error estimates for scalar conservation laws is constructed. In this paper, we focus on how the lack of consistency introduced by the nonuniformity of the grids influences the convergence of flux-splitting monotone schemes to the entropy solution. We obtain the optimal rate of convergence of $(\Delta x)^{1/2}$ in $L^\infty(L^1)$ for consistent schemes in arbitrary grids *without* the use of any regularity property of the approximate solution. We then extend this result to less consistent schemes, called $p-$consistent schemes, and prove that they converge to the entropy solution with the rate of $(\Delta x)^{\min\{1/2,p\}}$ in $L^\infty(L^1)$; again, no regularity property of the approximate solution is used. Finally, we propose a new explanation of the fact that even inconsistent schemes converge with the rate of $(\Delta x)^{1/2}$ in $L^\infty(L^1)$. We show that this well-known *supraconvergence* phenomenon takes place because the *consistency* of the numerical flux and the fact that the scheme is written in *conservation form* allows the regularity properties of its approximate solution (total variation boundedness) to compensate for its lack of consistency; the nonlinear nature of the problem does not play any role in this mechanism. All the above results hold in the multidimensional case, provided the grids are Cartesian products of one-dimensional nonuniform grids.

1. INTRODUCTION

This is the second of a series of papers in which we develop a theory of *a priori* error estimates, that is, estimates given solely in terms on the exact solution, for numerical methods for the scalar conservation law [11]

$$v_t + \nabla \cdot f(v) = 0, \quad \text{in } (0,T)\times\mathbb{R}^d, \tag{1.1a}$$

$$v(0) = v_0, \quad \text{on } \mathbb{R}^d. \tag{1.1b}$$

In the first paper of this series [4], we constructed a general approach aimed at obtaining *a priori* error estimates for numerical methods for scalar conservation laws

Received by the editor November 27, 1995 and, in revised form, May 6, 1996.
1991 *Mathematics Subject Classification.* Primary 65M60, 65N30, 35L65 .
Key words and phrases. A priori error estimates, irregular grids, monotone schemes, conservation laws, supraconvergence .
The first author was partially supported by the National Science Foundation (Grant DMS-9407952) and by the University of Minnesota Supercomputer Institute.
The second author was partially supported by the Army Research Office through grant DAAH04-95-1-0419.

by a suitable modification of Kuznetsov approximation theory [12]. We illustrated the approach by establishing optimal error estimates for the Engquist-Osher scheme [5] on one-dimensional uniform grids without using any smoothness property of the approximate solution generated by the scheme; in previous work, [2], [3], [13]–[16], [18], [19], [20], [22], [24], [26], regularity properties of the approximate solution were always used (see also [23]). The extension of this result to the case of nonuniform grids is by no means trivial since the nonuniformity of the grids introduces a "loss" of consistency (see, for example, Hoffman [9], Pike [21], and Turkel [25]) which, nevertheless, *does not* deteriorate the rate of convergence of the global error. This paper is devoted to the study of this *supraconvergence* phenomenon, that is, to the study of the relation between the part of the truncation error generated by the lack of consistency of the scheme and the global error.

Supraconvergence of numerical schemes has been analyzed in a variety of cases. For example, Manteuffel and White [17] studied *supraconvergence* for linear, second-order boundary value problems, Kreiss *et al.* [11] for high-order linear differential equations, B. Wendroff and A.B. White [27], [28] for nonlinear hyperbolic systems, García-Archilla and Sanz-Serna [7] for third-order finite differences, and García-Archilla [6] for the Korteweg-de Vries equation. To illustrate this *supraconvergence* phenomenon in our setting, let us consider the standard Engquist-Osher scheme on nonuniform grids, i.e.,

$$(u_j^{n+1} - u_j^n)/\Delta t + (f^{EO}(u_j^n, u_{j+1}^n) - f^{EO}(u_{j-1}^n, u_j^n))/\Delta_j = 0, \qquad n \in \mathbb{N}, j \in \mathbb{Z},$$

with numerical flux $f^{EO}(a,b) = f^+(a) + f^-(b)$, f^+ and f^- being respectively the increasing and decreasing part of f. As usual, $\Delta_j = x_{j+1/2} - x_{j-1/2}$ denotes the cell centered around the node x_j. Assuming that the solution v is smooth, the (formal) truncation error is given by $TE^f(t^n, x_j) = TE^f_{visc} + TE^f_{cons} + TE^f_{h.o.t}$, where

$$\begin{aligned}
TE^f_{visc} &= \frac{\Delta t}{2}\,\partial_x({f'}^2(v_j^n)\partial_x v_j^n) - \frac{\Delta_{j+1/2}^2 + \Delta_{j-1/2}^2}{4\Delta_j}\,\partial_x(|f'(v_j^n)|\partial_x v_j^n) \\
&\qquad + \frac{\Delta_{j+1/2}^2 - \Delta_{j-1/2}^2}{4\Delta_j}\,\partial_x(f'(v_j^n)\partial_x v_j^n), \\
TE^f_{cons} &= (\frac{\Delta_{j+1/2} + \Delta_{j-1/2}}{2\Delta_j} - 1)\, f'(v_j^n)\partial_x v_j^n + \frac{-\Delta_{j+1/2} + \Delta_{j-1/2}}{2\Delta_j}\,|f'(v_j^n)|\,\partial_x v_j^n, \\
TE^f_{h.o.t} &= \mathcal{O}(\frac{\Delta_{j+1/2}^3}{\Delta_j}) + \mathcal{O}(\frac{\Delta_{j-1/2}^3}{\Delta_j}) + \mathcal{O}(\Delta t^2),
\end{aligned}$$

where v_j^n stands for $v(t^n, x_j)$ and $\Delta_{j+1/2} = (\Delta_j + \Delta_{j+1})/2$. The above terms correspond respectively to the numerical viscosity of the scheme, to the consistency of the scheme, and to some "high-order" terms; note that the term TE^f_{cons} vanishes if uniform grids are considered. It is easy to see that the (formal) truncation error tends to zero upon refinement if Δ_j varies smoothly with respect to j. Convergence can thus reasonably be expected in this case. On the other hand, if nonsmooth grids are considered, the scheme is not consistent. Indeed if, for instance, the grids $\ldots, \Delta x/2, \Delta x, \Delta x/2, \Delta x, \ldots$ are considered, the term TE^f_{cons} does *not* tend to zero, and thus neither does the (formal) truncation error $TE^f(t^n, x_j)$.

The following numerical example shows that the inconsistency of the scheme on rough grids does not prevent it from converging at the optimal rate. In Figure 1,

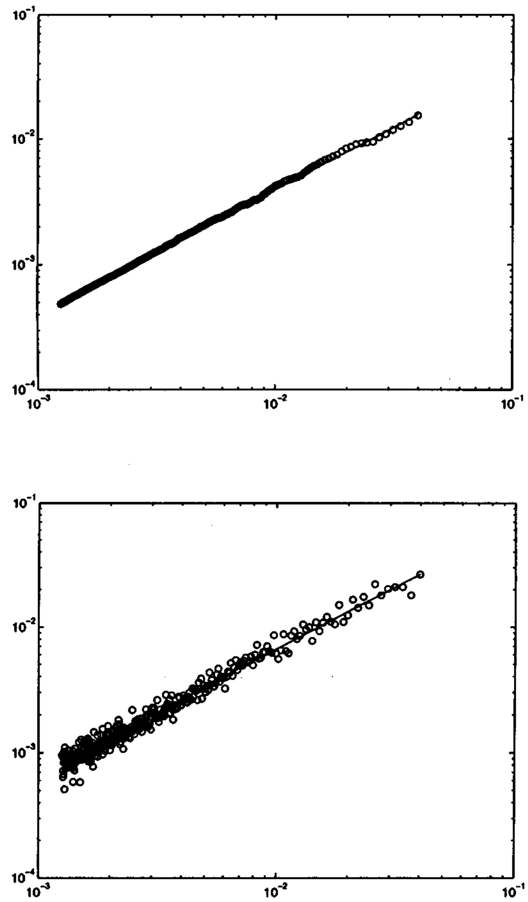

FIGURE 1. L^1-error vs. Δx for a continuous (top), and discontinuous (bottom) solution, on random grids

we display the performance of the Engquist-Osher scheme on the classical example of the Burgers' equation with periodic boundary conditions and a sinusoidal initial condition (see [8] for details).

About 400 randomly generated—and thus non smooth—grids were considered. The global L^1-error at the final time is represented with respect to Δx, size of the largest element. In Figure 1 (top), the exact solution is smooth; the convergence rate is one. In Figure 1 (bottom), the exact solution exhibits a discontinuity but, interestingly enough, the scheme converges without any loss in the numerical rate of convergence. This shows that the (formal) truncation error is a poor indicator of the quality of a numerical algorithm.

In this paper, we obtain the *proper* definition of the truncation error and show how to use it (i) to obtain *a priori* error estimates for flux-splitting monotone schemes in nonuniform grids, and (ii) to explain the *supraconvergence* phenomenon. Although Sanders [22] did establish an optimal error estimate for monotone schemes on nonuniform grids, his analysis relied on several regularity properties of the approximate solution, in particular total variation boundedness. This is a point of significant importance, if one recalls that even the simplest schemes, the monotone schemes, have not been proven to generate approximate solutions with this kind of regularity, when defined on general triangulations. In this paper, to obtain our *a priori* error estimates, we do not use any regularity property of the approximate solution; as a consequence, we are forced to use suitable definitions of consistency. Thus, we obtain the optimal rate of convergence of $(\Delta x)^{1/2}$ in $L^\infty(L^1)$, for consistent schemes in arbitrary grids. We also consider a class of numerical schemes of varying degree of consistency called $p-$consistent and prove that they converge to the entropy solution with the rate of $(\Delta x)^{\min\{1/2,p\}}$ in $L^\infty(L^1)$. In both cases, no regularity property of the approximate solution is used.

To explain the *supraconvergence* of the numerical schemes under consideration (which was proven by Sanders [22]) we allow ourselves to use the total variation boundedness of the approximate solution but *only* to estimate the term that appears in the *proper* truncation error due to the inconsistency introduced by the nonuniformity of the grids. We show that the optimal rate of convergence of $(\Delta x)^{1/2}$ in $L^\infty(L^1)$ can be obtained, even for inconsistent schemes, because the *consistency* of the numerical flux and the fact that the scheme is written in *conservation form* allow the regularity properties of the numerical approximation to compensate for the lack of consistency of the scheme; the nonlinearity of the problem does not play any role in this mechanism. To the knowledge of the authors, this is the first rigorous explanation of a supraconvergence phenomenon for hyperbolic problems with low regularity; the study of B. Wendroff and A.B. White [27], [28] on hyperbolic systems is formal and applies to smooth solutions only.

Finally, we strongly emphasize that, although all our results are stated and proved in a one-dimensional framework, they can be immediately extended to the case of multidimensional problems, provided the grids are *Cartesian products of nonuniform one-dimensional grids*. The case of time-varying meshes will not be considered in this paper since it would add a great deal of complexity to the already very technical analysis presented. To the authors' knowledge, no such result is available in the present context.

The paper is organized as follows. In §2, the numerical schemes under consideration are presented, related technical assumptions are discussed, and the main results are stated and discussed. In §3, we give a proof of our main result. Concluding remarks are offered in §4.

2. The numerical schemes and the main results

a. The numerical schemes. Given a partition of $\mathbb{R}^+$, $\{t^n = n\Delta t\}_{n\in\mathbb{N}}$, and a grid or partition of $\mathbb{R}$, $\{x_{j+1/2}\}_{j\in\mathbb{Z}}$, we define an approximation u to the entropy solution v of (1.1) (with $d=1$) as the piecewise-constant function

$$u(t,x) = u_j^n, \qquad \text{for } (t,x) \in [t^n, t^{n+1}) \times (x_{j-1/2}, x_{j+1/2}), \tag{2.1}$$

constructed as follows. At $t=0$, the degrees of freedom of u are given by

$$u_j^0 = \frac{1}{\Delta_j} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(s)\, ds. \tag{2.2a}$$

The remaining degrees of freedom are defined by the following flux-splitting scheme in *conservation form*:

$$(u_j^{n+1} - u_j^n)/\Delta t + (f_{j+1/2}^n - f_{j-1/2}^n)/\Delta_j = 0, \qquad n\in\mathbb{N}, j\in\mathbb{Z}, \tag{2.2b}$$

where $\Delta_j = x_{j+1/2} - x_{j-1/2}$ and the numerical flux $f^n_{j+1/2} = f_{j+1/2}(u^n_j, u^n_{j+1})$ has the form

$$f^n_{j+1/2} = f^n_{cent,j+1/2} - f^n_{visc,j+1/2}, \tag{2.3a}$$

with

$$f^n_{cent,j+1/2} = \frac{a_{j+1/2}}{\Delta_{j+1/2}}\, f(u^n_j) + \frac{b_{j+1/2}}{\Delta_{j+1/2}}\, f(u^n_{j+1}), \tag{2.3b}$$

and

$$f^n_{visc,j+1/2} = \frac{\alpha_{j+1/2}}{\Delta_{j+1/2}} \left(N(u^n_{j+1}) - N(u^n_j) \right), \tag{2.3c}$$

where $\Delta_{j+1/2} = (\Delta_j + \Delta_{j+1})/2$. We assume that the flux $f^n_{j+1/2}$ is *consistent* with the nonlinearity f, i.e., that $f_{j+1/2}(u,u) = f(u)$; this is equivalent to assume

$$a_{j+1/2} + b_{j+1/2} = \Delta_{j+1/2}. \tag{2.3d}$$

We also require

$$\max\{a_{j+1/2}, b_{j+1/2}, 0\} \le \alpha_{j+1/2} \le \Delta x \equiv \sup_{j\in\mathbb{Z}} \Delta_j, \tag{2.3e}$$

and

$$N'(s) \ge |\, f'(s)\,|. \tag{2.3f}$$

Two standard examples of viscosity N are $N(u) = \int^u |\, f'(s)\,|\, ds$ (Engquist-Osher flux) and $N(u) = Cu$ (Lax-Friedrichs flux), where C is chosen as to satisfy (2.3f). As is well-known, condition (2.3f) ensures the *monotonicity* of the scheme under a suitable condition on the size of Δt which in our case turns out to be the following:

$$\frac{\Delta t}{\Delta_j} \|\, N'(u)\,\| \le \mathrm{cfl}(j), \qquad j \in \mathbb{Z}, \tag{2.4a}$$

where

$$cfl^{-1}(j) = \Big|\, \frac{a_{j+1/2}}{\Delta_{j+1/2}} - \frac{b_{j-1/2}}{\Delta_{j-1/2}}\,\Big| + \Big(\frac{\alpha_{j+1/2}}{\Delta_{j+1/2}} + \frac{\alpha_{j-1/2}}{\Delta_{j-1/2}}\Big), \tag{2.4b}$$

$$\|\, N'(u)\,\| = \sup_{\substack{t\in(0,T)\\ x\in\mathbb{R}}} N'(u(t,x)). \tag{2.4c}$$

Note that (for the Engquist-Osher scheme, for example) the above stability condition (2.4) boils down to the usual $\mathrm{cfl}(j) \equiv 1$ in the case of uniform grids. It should also be noted that (2.4) is essentially a condition of the type $\Delta t \le k \inf_{j\in\mathbb{Z}} \Delta_j$, where k is a positive constant independent of the grid. It is therefore the smallest element Δ_0 which limits the size of the time step Δt. Although customary, such a condition might be unreasonably stringent for most of the elements. An interesting alternative is discussed in [1].

b. Consistency and *a priori* error estimates. Before stating our error estimates, we need to elaborate on the notion of consistency of the schemes.

The (formal) truncation error $TE^f(t^n, x_j)$ for the schemes under consideration can be split into three terms, $TE^f_{visc} + TE^f_{cons} + TE^f_{h.o.t.}$, defined as follows:

$$
\begin{aligned}
TE^f_{visc} =& \frac{\Delta t}{2}\,\partial_x({f'}^2(v^n_j)\partial_x v^n_j) - \frac{\Delta_{j-1/2}\,\alpha_{j-1/2} + \Delta_{j+1/2}\,\alpha_{j+1/2}}{2\Delta_j}\,\partial_x(N'(v^n_j)\partial_x v^n_j)\\
&+ \frac{b_{j+1/2}\,\Delta_{j+1/2} - a_{j-1/2}\,\Delta_{j-1/2}}{2\Delta_j}\,\partial_x(f'(v^n_j)\partial_x v^n_j),\\
TE^f_{cons} =& \frac{b_{j+1/2} + a_{j-1/2} - \Delta_j}{\Delta_j}\;\partial_x f(v^n_j) - \frac{\alpha_{j+1/2} - \alpha_{j-1/2}}{\Delta_j}\,\partial_x N(v^n_j),\\
TE^f_{h.o.t.} =& \mathcal{O}(\Delta t^2) + r_j\,\mathcal{O}(\Delta x^2),
\end{aligned}
$$

where $r_j = \max\{\frac{\Delta_{j-1}}{\Delta_j}, \frac{\Delta_{j+1}}{\Delta_j}\}$. Note that we can use the *consistency* of the numerical flux (2.3) to write

$$TE^f_{cons} = -\frac{\hat{\delta}_{j+1/2} - \hat{\delta}_{j-1/2}}{\Delta_j}\;\partial_x f(v^n_j) - \frac{\alpha_{j+1/2} - \alpha_{j-1/2}}{\Delta_j}\,\partial_x N(v^n_j),$$

where

$$\hat{\delta}_{j+1/2} = a_{j+1/2} - \frac{1}{2}\Delta_{j+1}.$$

Although it is not clear at this point, it is this *structure* of the consistency error TE^f_{cons} which allows the phenomenon of *supraconvergence* to take place. What is clear, however, is that the consistency error is identically zero if both $\hat{\delta}_{j+1/2}$ and $\alpha_{j+1/2}$ are constant. It is thus reasonable to measure the degree of *consistency* of the scheme by some seminorm related to the variation of $\hat{\delta}$ and α.

Our analysis shows that the correct quantity to consider is not $\hat{\delta}_{j+1/2}$ but

$$\delta_{j+1/2} = a_{j+1/2} - \frac{1}{2}\Delta_j, \tag{2.5}$$

and that the consistency of the error should be measured with the following seminorm:

$$|\,\zeta\,|_{var,1/2} = \sup_{x\in\mathbb{R}} \frac{\sum_{|x_j - x|\le(\Delta x)^{1/2}} |\,\zeta_{j+1/2} - \zeta_{j-1/2}\,|}{(\Delta x)^{1/2}}. \tag{2.6a}$$

This motivates the following concepts of consistency. We say that the scheme is $p-$consistent with respect to the family of grids $\big\{\,\{\,x_{j+1/2}\,\}_{j\in\mathbb{Z}}\,\big\}_{\Delta x>0}$ if there are two nonnegative constants C_δ and C_α such that

$$|\,\delta\,|_{var,1/2} \le C_\delta\,(\Delta x)^p, \quad |\,\alpha\,|_{var,1/2} \le C_\alpha\,(\Delta x)^p. \tag{2.6b}$$

If $C_\delta = C_\alpha = 0$, we say that the scheme is consistent. For example, for the one-parameter family of schemes

$$a_{j+1/2} = \frac{1}{2}\big((1-\theta)\Delta_j + \theta\Delta_{j+1}\big),$$

$$b_{j+1/2} = \frac{1}{2}\big((1-\theta)\Delta_{j+1} + \theta\Delta_j\big), \quad \alpha_{j+1/2} = \frac{1}{2}\Delta x,$$

for $\theta \in [0,1]$, we have $\delta_{j+1/2} = \theta\,(\Delta_{j+1} - \Delta_j)/2$. Moreover, it is clear that the schemes are consistent for $\theta = 0$ regardless of the family of grids. For $\theta \in (0,1]$, these schemes are $p-$consistent if the grids are such that

$$\sum_{|x_j - x| \leq (\Delta x)^{1/2}} |\,\Delta_{j-1} - 2\Delta_j + \Delta_{j+1}\,| \leq (\Delta x)^{1/2} \cdot C_\delta\,(\Delta x)^p.$$

This property holds if the grids are $p-$smooth, that is, if there is a constant κ such that

$$\big|r_h - 1\big| \leq \kappa\,\Delta x^p, \qquad r_h = \sup_{j\in\mathbb{Z}} \frac{\Delta_{j\pm 1}}{\Delta_j}. \tag{2.7}$$

Note that for $0-$smooth grids like $\dots, \Delta x/2, \Delta x, \Delta x/2, \Delta x, \dots$, the schemes above are $0-$consistent and clearly inconsistent, except for the scheme obtained with $\theta = 0$.

We are now ready to state our error estimate which, following [4], is expressed in terms of the numerical viscosity associated to the scheme under consideration and in terms of the measure of consistency introduced above.

Theorem 2.1. *Let the Courant-Friedrichs-Levy condition* (2.4) *be satisfied. Let u be the piecewise-constant solution given by the scheme* (2.2) *with coefficients satisfying* (2.3)*, let v be the entropy solution, and set $\mathcal{R}(v_0) = [\inf_{x\in\mathbb{R}} v_0(x), \sup_{x\in\mathbb{R}} v_0(x)]$. Then*

$$\begin{aligned}\|\,u(t^N) - v(t^N)\,\|_{L^1(\mathbb{R})} \leq{}& 2\,\|\,u_0 - v_0\,\|_{L^1(\mathbb{R})} + 8\,|\,v_0\,|_{TV(\mathbb{R})}\sqrt{2t^N\,\|\,\nu_v\,\|}\,(\Delta x)^{1/2}\\ &+ C\,|\,v_0\,|_{TV(\mathbb{R})}\,\big(\,|\,\delta\,|_{var,1/2} + |\,\alpha\,|_{var,1/2}\big)\\ &+ |\,v_0\,|_{TV(\mathbb{R})}\,(\,b_1(\Delta x)^{3/4} + b_2\,\Delta x\,),\end{aligned}$$

where $\|\,\nu_v\,\| = \sup_{j\in\mathbb{Z}}\sup_{w\in\mathcal{R}(v_0)} \nu_j(w)$ and the local viscosity coefficient ν_j is given by

$$\begin{aligned}\nu_j(w) = \frac{1}{2}\bigg|&\bigg(\frac{\alpha_{j+1/2}}{\Delta x}\frac{\Delta_j + 2\Delta_{j+1}}{3\Delta_j} + \frac{\alpha_{j-1/2}}{\Delta x}\frac{\Delta_j + 2\Delta_{j-1}}{3\Delta_j}\bigg)N'(w)\\ +&\bigg(\frac{a_{j+1/2}}{\Delta x}\frac{\Delta_j + 2\Delta_{j+1}}{3\Delta_j} - \frac{b_{j-1/2}}{\Delta x}\frac{\Delta_j + 2\Delta_{j-1}}{3\Delta_j}\bigg)f'(w) - \frac{\Delta t}{\Delta x}(f'(w))^2\bigg|.\end{aligned}$$

The constant C is given by

$$C = 4\,\|\,N'(v)\,\|\,(t^N + \sqrt{2t^N/\|\,\nu_v\,\|}\,)\,(1 + b_0\,(\Delta x)^{1/4}),$$

and the constants b_0, b_1, and b_2 are locally bounded functions that depend solely on the quantities $\|\,f'(v)\,\|\Delta t/\Delta x$, $\|\,f'(v)\,\|/\|\,\nu_v\,\|$, and $\{t^N\,\|\,\nu_v\,\|\}^{1/2}$. Moreover, if the entropy solution has a finite number of discontinuities on each compact set of $(0,T)\times\mathbb{R}$, we can take $\|\,\nu_v\,\| = \sup_{\substack{t\in(0,\tau)\\ x\in\mathbb{R}}} \nu_j(v(t,x-0), v(t,x+0))$, where

$\nu_j(v^-,v^+) \equiv \sup_{u\in[v^-\wedge v^+, v^-\vee v^+]} |\nu_j(u;v^-,v^+)|$ *and*

$$\begin{aligned}\nu_j(u;v^-,v^+) =& \frac{1}{2}\Bigg\{ \left(\alpha_{j+1/2}\frac{\Delta_j+2\Delta_{j+1}}{3\Delta_j} + \alpha_{j-1/2}\frac{\Delta_j+2\Delta_{j-1}}{3\Delta_j}\right)\frac{[\mathcal{N}(v)]}{[v]} \\ &+ \left(a_{j+1/2}\frac{\Delta_j+2\Delta_{j+1}}{3\Delta_j} - b_{j-1/2}\frac{\Delta_j+2\Delta_{j-1}}{3\Delta_j}\right)\frac{[F(v)]}{[v]} \\ &- \Delta t\frac{[F(v)]}{[v]}\frac{[f(v)]}{[v]}\Bigg\}.\end{aligned}$$

In the above expression, $[v] = v^+ - v^-$, $[F(v)] = \int_{v^-}^{v^+} f'(s)\, sgn(s-u)\, ds$, *and* $[\mathcal{N}(v)] = \int_{v^-}^{v^+} N'(s)\, sgn(s-u)\, ds$.

An immediate consequence of this result is the following.

Corollary 2.2 ($p-$consistent schemes). *With the notation and under the assumptions of Theorem* 2.1, *if the scheme is $p-$consistent, we have*

$$\begin{aligned}\| u(t^N) - v(t^N)\|_{L^1(\mathbb{R})} \le 2\,\| u_0 - v_0\|_{L^1(\mathbb{R})} + 8\,| v_0|_{TV(\mathbb{R})}\sqrt{2t^N\,\|\nu_v\|}\,(\Delta x)^{1/2} \\ + \mathcal{O}((\Delta x)^{\min\{p,3/4\}}).\end{aligned}$$

For consistent schemes, we have that $\delta_{j+1/2}\equiv\delta$, $\alpha_{j+1/2}\equiv\alpha$, and we can write

$$\nu_j(w) \le \left| \frac{\alpha+|\delta|}{\Delta x}\,N'(w) - \frac{\Delta t}{2\,\Delta x}\,(f'(w))^2\right| + \Theta(w),$$

where

$$\begin{aligned}\Theta(w) \le& \left(\frac{2}{3}\Big(\frac{\alpha}{\Delta x} + \frac{|\delta|}{\Delta x}\Big)\,|\,r_h - 1\,| + \Big(\frac{10}{3}+\frac{8}{3}|\,r_h-1\,|\Big)\,|\,r_h - 1\,|\right) N'(w) \\ \le& \,C\,N'(w)\,(\Delta x)^q,\end{aligned}$$

for $q-$smooth grids. Thus, Theorem 2.1 gives the following result.

Corollary 2.3 (Consistent schemes). *With the notation and under the assumptions of Theorem* 2.1, *if the grids are $q-$smooth and if the scheme is consistent, we have*

$$\begin{aligned}\| u(t^N) - v(t^N)\|_{L^1(\mathbb{R})} \le 2\,\| u_0 - v_0\|_{L^1(\mathbb{R})} + 8\,| v_0|_{TV(\mathbb{R})}\sqrt{2t^N\,\|\nu_v\|}\,(\Delta x)^{1/2} \\ + \mathcal{O}((\Delta x)^{\min\{1/2+q,3/4\}}),\end{aligned}$$

where

$$\|\nu_v\| = \sup_{w\in\mathcal{R}(v_0)} \left| \frac{\alpha+|\delta|}{\Delta x}\,N'(w) - \frac{\Delta t}{2\Delta x}\,(f'(w))^2\right|.$$

Note that even for $0-$smooth grids, the optimal rate of convergence of $\mathcal{O}((\Delta x)^{1/2})$ is achieved by the above schemes.

c. Sketch of the proof. In what follows, we give an overview of the proof of Theorem 2.1 which is given in full detail in §3. We start with the following approximation inequality [4, Proposition 7.6]. If $e(t^n)$ denotes the error $\| u(t^n) - v(t^n) \|_{L^1(\mathbb{R})}$, then

$$\begin{aligned} e(t^N) \leq 2\, e(0) + 8\, \left(\epsilon_x + \epsilon_t \|f'(v)\|\right) |\, v_0 \,|_{TV(\mathbb{R})} + 2 \,\|\, f'(v) \,\|\, |\, v_0 \,|_{TV(\mathbb{R})}\, \Delta t \\ + 2 \lim_{w \to \chi} \sup_{1 \leq n \leq N} \left\{ E^{\star}_{div}(u, v; t^n)/W(t^n) - E_{diss}(u_h, v; t^n)/W(t^n) \right\}, \end{aligned}$$

where the so-called dual form $E^{\star}_{div}(u, v; t^n)$ is, in this case, nothing but the truncation error and the form $E_{diss}(u_h, v; t^n)$ contains the information on the entropy dissipation (or "hyperbolic coercivity") of the numerical scheme. The third term in the right-hand side reflects the fact that the scheme is first-order accurate in time. The parameters ϵ_x and ϵ_t are auxiliary positive numbers that will be suitably chosen after obtaining the estimates of the forms $E^{\star}_{div}(u, v; t^n)$ and $E_{diss}(u_h, v; t^n)$. The functions ω, χ, and W are auxiliary functions to be precisely defined in §3.a.

Since the numerical schemes under consideration are monotone, it can be easily proven that

$$-E_{diss}(u_h, v; t^N) \leq 0,$$

under the condition (2.4) on the size of Δt.

To estimate the dual form $E^{\star}_{div}(u, v; t^n)$, we first show that it is bounded by the truncation error

$$E^{\star}_{div}(u, v; t^n) \leq TE(u, v; t^n),$$

and then we obtain the corresponding estimate.

To illustrate the estimate of $E^{\star}_{div}(u, v; t^n)$, let us consider that both the entropy solution v and the "approximate solution" u are smooth. We also assume that the functions a, b, δ, and α defining the coefficients of the numerical scheme are smooth functions. In this case, the truncation error $TE = TE(u, v; T)$ can be written as the sum of the following three terms

$$\begin{aligned} TE_{visc} &= \int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} \Delta x\, V(u, v; x')\, \varphi_{xx}\, dx'\, dt'\, dx\, dt, \\ TE_{cons} &= - \int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} \left\{ \delta_{x'}(x')\, F(u, v) + \alpha_{x'}(x')\, \mathcal{N}(u, v) \right\} \varphi_x\, dx'\, dt'\, dx\, dt, \\ TE_{h.o.t} &= \int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} \frac{(\Delta t)^2}{6} \mathcal{F}(u, v)\, \varphi_{xxt}\, dx'\, dt'\, dx\, dt \\ &\quad + \int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} \left\{ \mathcal{P}(x')\, F(u, v) + \mathcal{Q}(x')\, \mathcal{N}(u, v) \right\} \varphi_{xxx}\, dx'\, dt'\, dx\, dt, \end{aligned}$$

where, in order to render as clear as possible the manipulations that will be performed, we abbreviated $v(t, x)$ by v, $u(t', x')$ by u, and the auxiliary function $\varphi(t, x, t', x')$ by φ. The functions V, F, $\mathcal{N}$ and $\mathcal{F}$ are related to the numerical viscosity coefficient

$$\nu(w; x') = \frac{\alpha(x')}{\Delta x}\, N'(w) + \frac{b(x') - a(x')}{2\, \Delta x}\, f'(w) - \frac{\Delta t}{2\, \Delta x}\, (f'(w))^2,$$

and to the functions f and N as follows:

$$V(u,v;x') = \int_v^u \nu(s;x')\,U'(u-s)\,ds, \quad F(u,v) = \int_v^u f'(s)\,U'(u-s)\,ds,$$
$$\mathcal{N}(u,v) = \int_v^u N'(s)\,U'(u-s)\,ds, \quad \mathcal{F}(u,v) = \int_v^u (f'(s))^2\,U'(u-s)\,ds,$$

where $U(w) = |\,w\,|$. The functions $\mathcal{P}$ and $\mathcal{Q}$ satisfy

$$\|\,\mathcal{P}\,\|_{L^\infty(\mathbb{R})}, \|\,\mathcal{Q}\,\|_{L^\infty(\mathbb{R})} \le (\Delta x)^2/2.$$

Before estimating the truncation error TE, let us compare it with the (formal) truncation error TE^f, which is the sum of the following terms:

$$\begin{aligned} TE^f_{visc}(t,x) &= \frac{1}{2}\Delta t\,\partial_x(\,{f'}^2(v(t,x))\,\partial_x v(t,x)) - \alpha(x)\,\partial_{xx}(N(v(t,x))) \\ &\quad + \frac{1}{2}(b(x)-a(x))\,\partial_{xx}(f(v(t,x))), \\ TE^f_{cons}(t,x) &= -\big\{\delta_x(x)\,\partial_x f(v(t,x)) + \alpha_x(x)\,\partial_x N(v(t,x))\big\}, \\ TE^f_{h.o.t.} &= \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2). \end{aligned}$$

We see that the definition of the (formal) truncation error TE^f collapses when v is a nonsmooth function. However, the truncation error TE remains defined even if v and u are only bounded and measurable. Moreover, in the expression of the truncation error TE, it is possible to integrate by parts very easily due to the fact that the functions $v = v(t,x)$ and $u = u(t',x')$ are always evaluated at different points; this key feature was introduced by Kružkov [11]. In order to compensate for this "doubling of the variables," the auxiliary function φ is introduced and is defined to be an approximation of the product of the Dirac delta functions with support $\{t = t'\}$ and $\{x = x'\}$ respectively; more precisely, $\varphi(t,x,t',x') = \{w((t-t')/\epsilon_t)/\epsilon_t\}\,\{\eta((x-x')/\epsilon_x)/\epsilon_x\}$, where w and η are positive, even, smooth functions of unit mass and support in $[-1,1]$.

We are now ready to estimate TE. To estimate TE_{visc}, we integrate by parts in the variable x and use the definition of the function $V(u,v;x')$ to obtain

$$\begin{aligned} TE_{visc} &= \int_0^T\!\!\int_{\mathbb{R}}\int_0^T\!\!\int_{\mathbb{R}} \Delta x\,\nu(v;x')\,v_x\,U'(u-v)\,\varphi_x\;dx'\,dt'\,dx\,dt \\ &\le \Delta x\,\|\,\nu_v\,\|\int_0^T\!\!\int_{\mathbb{R}} |\,v_x\,| \left\{\int_0^T\!\!\int_{\mathbb{R}} |\,\varphi_x\,|\;dx'\,dt'\right\} dx\,dt \\ &\le 2\,C_0\,\frac{|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_x}\,\|\,\nu_v\,\|\Delta x, \end{aligned}$$

where $C_0 = T\,|v_0|_{TV(\mathbb{R})}\,W(T)$, and where $W(s)$ is the antiderivative of $w(s/\epsilon_t)/\epsilon_t$, since

$$\int_0^T\!\!\int_{\mathbb{R}} |\,\varphi_x\,|\;dx'\,dt'\,dx\,dt \le 2\,\frac{|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_x}\,W(T), \quad \int_0^T\!\!\int_{\mathbb{R}} |\,v_x\,|\,dx\,dt \le\; T\,|v_0|_{TV(\mathbb{R})}.$$

To estimate the consistency truncation error, TE_{cons}, we integrate once again by parts in the variable x and use the definitions of the functions $F(u,v)$ and $\mathcal{N}(u,v)$:

$$\begin{aligned}
TE_{cons} &= -\int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} \left\{ \delta_{x'}(x')\, F(u,v) + \alpha_{x'}(x')\, \mathcal{N}(u,v) \right\} \varphi_x \, dx'\, dt'\, dx\, dt \\
&= -\int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} \left\{ \delta_{x'}(x')\, f'(v) + \alpha_{x'}(x')\, N'(v) \right\} U'(u-v)\, v_x\, \varphi \, dx'\, dt'\, dx\, dt \\
&\le \int_0^T\int_{\mathbb{R}} |\, f'(v)\,|\,|\, v_x\,| \left\{ \int_0^T\int_{\mathbb{R}} \varphi\, |\, \delta_{x'}(x')\,|\, dx'\, dt' \right\} dx\, dt \\
&\quad + \int_0^T\int_{\mathbb{R}} |\, N'(v)\,|\,|\, v_x\,| \left\{ \int_0^T\int_{\mathbb{R}} \varphi\, |\, \alpha_{x'}(x')\,|\, dx'\, dt' \right\} dx\, dt \\
&\le 2\, C_1 \,\Big(1 + \frac{(\Delta x)^{1/2}}{\epsilon_x} \Big) \,\|\, \eta\,\|_{L^\infty(\mathbb{R})} \big(|\,\delta\,|_{var,1/2} + |\,\alpha\,|_{var,1/2} \big),
\end{aligned}$$

where $C_1 = C_0\, \|N'(v)\|$, since

$$\int_0^T\int_{\mathbb{R}} \varphi\, |\, \zeta_{x'}(x')\,|\, dx'\, dt' \le 2\; W(T) \Big(1 + \frac{(\Delta x)^{1/2}}{\epsilon_x} \Big) \,\|\, \eta\,\|_{L^\infty(\mathbb{R})}\, |\, \zeta\,|_{var,1/2}.$$

Finally, to estimate $TE_{h.o.t}$, we integrate by parts in x, use the definitions of $F(u,v)$, $\mathcal{N}(u,v)$, and $\mathcal{F}$ and proceed as before to get

$$TE_{h.o.t} \le C_1 \left\{ \frac{(\Delta t)^2\, |\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_t\, \epsilon_x} \,\|\, f'\,\| + 2\, \frac{(\Delta x)^2\, |\,\eta'\,|_{TV(\mathbb{R})}}{\epsilon_x^2} \right\}.$$

Now, we pick η such that:

$$|\,\eta\,|_{TV(\mathbb{R})} = 1+\epsilon, \quad |\,\eta'\,|_{TV(\mathbb{R})} = 2+\epsilon+1/\epsilon, \quad \|\,\eta\,\|_{L^\infty(\mathbb{R})} = (1+\epsilon)/2,$$

and insert the above estimates into the right-hand side of the approximation inequality. To prove Theorem 2.1, we simply have to minimize the right-hand side of the approximation inequality with respect to the parameters ϵ_x, ϵ_t, and ϵ. It turns out that the optimal parameters are $\epsilon_x = \mathcal{O}((\Delta x)^{1/2})$, $\epsilon_t = \mathcal{O}((\Delta x)^{3/4})$, and $\epsilon = \mathcal{O}((\Delta x)^{1/4})$. The estimates of the truncation errors then take the form

$$\begin{aligned}
TE_{visc}(u,v;T)/W(T) &\le C_0'\,(\Delta x)^{1/2}, \\
TE_{cons}(u,v;T)/W(T) &\le C_1' \big(|\,\delta\,|_{var,1/2} + |\,\alpha\,|_{var,1/2} \big), \\
TE_{h.o.t.}(u,v;T)/W(T) &\le C_2'\,(\Delta x)^{3/4},
\end{aligned}$$

where the constants C_i', $i = 0,1,2$, are independent of Δx for Δx small enough.

d. An explanation of the *supraconvergence*. To illustrate the idea that allows the *supraconvergence* phenomenon to take place, we only need to show how to exploit the structure of the term TE_{cons} to obtain a better estimate. Since both terms of TE_{cons} are similar in structure, we concentrate only on the first:

$$\Theta = -\int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} \delta_{x'}(x')\, F(u,v)\varphi_x \, dx'\, dt'\, dx\, dt.$$

Note that if we do not want to use the variation of δ to estimate Θ, we can exploit the fact that it is possible to integrate by parts, this time with respect to x', to get an estimate involving a bound on the L^∞-norm of δ only. It is this structure of the consistency error (which, as we saw in §2.a, is a reflection of the *consistency* of the numerical flux and the *conservativity* of the scheme) which allows the phenomenon of *supraconvergence* to take place. The price to pay, however, is that we must give up the restriction of not using regularity properties of the approximate solution u, as we show next.

Thus, to estimate Θ, we integrate by parts in the variable x' and use the definition of the function $F(u,v)$ to obtain

$$\begin{aligned}\Theta &= -\int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} F(u,v)\,\varphi_x\,(\delta(x')-\bar{\delta})_{x'}\,dx'\,dt'\,dx\,dt \\ &= \int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} \left\{U'(u-v)\,f'(u)\,u_{x'}\varphi_x + F(u,v)\,\varphi_{x\,x'}\right\}\left\{\delta(x')-\bar{\delta}\right\}dx'\,dt'\,dx\,dt \\ &= \int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} U'(u-v)\left\{f'(u)\,u_{x'}\varphi_x + f'(v)\,v_x\,\varphi_{x'}\right\}\left\{\delta(x')-\bar{\delta}\right\}dx'\,dt'\,dx\,dt \\ &\le \int_0^T\int_{\mathbb{R}} |\,f'(u)\,|\,|\,u_{x'}\,|\left\{\int_0^T\int_{\mathbb{R}} |\,\varphi_x\,|\left\{\delta(x')-\bar{\delta}\right\}dx\,dt\right\}dx'\,dt' \\ &\quad + \int_0^T\int_{\mathbb{R}} |\,f'(v)\,|\,|\,v_x\,|\left\{\int_0^T\int_{\mathbb{R}} |\,\varphi_{x'}\,|\left\{\delta(x')-\bar{\delta}\right\}dx'\,dt'\right\}dx\,dt.\end{aligned}$$

At this point, it becomes clear that in order to estimate Θ, we must obtain a bound on the L^∞-norm of u and on its total variation. Since u is the "approximate solution" of a monotone scheme in one-space dimension, it is well-known that we have

$$\|\,f'(u)\,\|\,\|\,u\,\|_{L^\infty(0,t^N;TV(\mathbb{R}))} \le \|\,f'(v)\,\|\,|\,v_0\,|_{TV(\mathbb{R})},$$

With the above regularity property of the "approximate solution", we can obtain

$$\Theta \le 4\,C_1\,\frac{\|\,\delta\,\|_{L^\infty(\mathbb{R})/\mathbb{R}}}{\epsilon_x}\,|\,\eta\,|_{TV(\mathbb{R})},$$

where $\|\,\delta\,\|_{L^\infty(\mathbb{R})/\mathbb{R}} = \inf_{\bar{\delta}\in\mathbb{R}}\|\,\delta-\bar{\delta}\,\|_{L^\infty(\mathbb{R})}$, since

$$\int_0^T\int_{\mathbb{R}} |\,\varphi_{x'}\,|\,dx'\,dt' = \int_0^T\int_{\mathbb{R}} |\,\varphi_x\,|\,dx\,dt \le 2\,\frac{|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_x}\,W(T).$$

This implies the following upper bound for the consistency error:

$$TE_{cons}(u,v;T) \le 4\,C_1\,|\,\eta\,|_{TV(\mathbb{R})}\,\frac{\|\,\delta\,\|_{L^\infty(\mathbb{R})/\mathbb{R}} + \|\,\alpha\,\|_{L^\infty(\mathbb{R})/\mathbb{R}}}{\epsilon_x}.$$

The error estimate follows as in the previous section. A discrete version of the above argument can be easily obtained which, under the notation and assumptions of Theorem 2.1, leads to the error estimate:

$$\begin{aligned}\|\,u(t^N)-v(t^N)\,\|_{L^1(\mathbb{R})} &\le 2\,\|\,u_0-v_0\,\|_{L^1(\mathbb{R})} \\ &\quad + 8\,|\,v_0\,|_{TV(\mathbb{R})}\sqrt{2\,t^N\left(\|\,\nu_v\,\|\,\Delta x + 2\|\,N'(v)\,\|\left(\|\,\delta\,\|_{L^\infty(\mathbb{R})/\mathbb{R}} + \|\,\alpha\,\|_{L^\infty(\mathbb{R})/\mathbb{R}}\right)\right)} \\ &\quad + \left(\,b_1(\Delta x)^{3/4} + b_2\,\Delta x\,\right)|\,v_0\,|_{TV(\mathbb{R})},\end{aligned}$$

which gives the optimal rate of convergence of $(\Delta x)^{1/2}$, as expected. Although the above error estimate is new, we are more interested in the technique to obtain it since it sheds light into the *supraconvergence* phenomenon. As we have just shown, the optimal rate of convergence of $(\Delta x)^{1/2}$ can be obtained even though the scheme is not consistent because the *consistency* of its numerical flux and its *conservativity* makes possible for the lack of consistency of the scheme to be compensated by the regularity of its approximate solution u. The fact that the problem is nonlinear does not play any fundamental role in this mechanism.

3. Proof of Theorem 2.1

In this section, we prove our Theorem 2.1. This section is closely related to section 7 of [4] in which we establish the same result for the Engquist-Osher scheme defined in uniform grids. Thus, we shall use the same notation and omit detailed proofs when those are variations of similar proofs in [4].

a. The approximation inequality. We start by displaying the following inequality proven in [4, Proposition 7.6]. If $e(t^n)$ denotes the error $\| u(t^n) - v(t^n) \|_{L^1(\mathbb{R})}$, then

$$\begin{aligned} e(t^N) \leq 2\, e(0) + 8\, \left(\epsilon_x + \epsilon_t \|f'(v)\|\right) |\, v_0 \,|_{TV(\mathbb{R})} + 2 \, \| \, f'(v) \, \| \, |\, v_0 \,|_{TV(\mathbb{R})}\, \Delta t \\ + 2 \lim_{w \to \chi} \sup_{1 \leq n \leq N} \left\{ E^{\star}_{div}(u, v; t^n)/W(t^n) - E_{diss}(u_h, v; t^n)/W(t^n) \right\}, \end{aligned}$$

where, in this case, the form $E_{diss}(u_h, v; t^N)$ is given by

$$E_{diss}(u_h, v; t^N) = \int_0^{t^N} \int_{\mathbb{R}} \sum_{n=0}^{N-1} \sum_{j \in \mathbb{Z}} LRED_j^n(v(t,x))\, \phi(t,x,t^{n+1},x_j)\, \Delta_j\, \Delta t\, dx\, dt,$$

where the local rate of entropy dissipation $LRED_j^n(c)$ is given by

$$\begin{aligned} LRED_j^n(c) &= \frac{1}{\Delta t} \int_{u_j^{n+1}}^{u_j^n} (p_j^1(u_j^n) - p_j^1(s))\, U''(s-c)\, ds \\ &\quad + \frac{1}{\Delta_j} \int_{u_j^{n+1}}^{u_{j-1}^n} (p_j^2(u_{j-1}^n) - p_j^2(s))\, U''(s-c)\, ds \\ &\quad + \frac{1}{\Delta_j} \int_{u_j^{n+1}}^{u_{j+1}^n} (p_j^3(u_{j+1}^n) - p_j^3(s))\, U''(s-c)\, ds, \\ p_j^1 &= s - \frac{\Delta t}{\Delta_j} \left(\frac{a_{j+1/2}}{\Delta_{j+1/2}} - \frac{b_{j-1/2}}{\Delta_{j-1/2}}\right) f(s) - \frac{\Delta t}{\Delta_j} \left(\frac{\alpha_{j+1/2}}{\Delta_{j+1/2}} + \frac{\alpha_{j-1/2}}{\Delta_{j-1/2}}\right) N(s), \\ p_j^2 &= \quad \frac{a_{j-1/2}}{\Delta_{j-1/2}}\, f(s) + \frac{\alpha_{j-1/2}}{\Delta_{j-1/2}}\, N(s), \\ p_j^3 &= -\frac{b_{j+1/2}}{\Delta_{j+1/2}}\, f(s) + \frac{\alpha_{j+1/2}}{\Delta_{j+1/2}}\, N(s), \end{aligned}$$

and the dual form $E^{\star}_{div}(u_h, v; t^N)$ is given by

$$\begin{aligned}
E^{\star}_{div}(u_h, v; t^N) = &- \int_0^{t^N}\int_{\mathbb{R}}\int_0^{t^N}\int_{\mathbb{R}} U(u(t',x') - v(t,x))\,\varphi_t(t,x,t',x')\,dx\,dt\,dx'\,dt' \\
&+ \int_0^{t^N}\int_{\mathbb{R}}\int_{\mathbb{R}} U(u(t',x') - v(t^N,x))\,\varphi(t^N,x,t',x')\,dx\,dt\,dx' \\
&- \int_0^{t^N}\int_{\mathbb{R}}\int_{\mathbb{R}} U(u(t',x') - v_0(x))\,\varphi(0,x,t',x')\,dx\,dt\,dx' \\
&- \sum_{n=0}^{N-1}\sum_{j\in\mathbb{Z}}\int_0^{t^N}\int_{\mathbb{R}} \Big\{ a_{j+1/2}\frac{\phi(t,x,t^{n+1},x_j) - \phi(t,x,t^{n+1},x_{j+1})}{\Delta_{j+1/2}} \\
&\qquad + b_{j-1/2}\frac{\phi(t,x,t^{n+1},x_{j-1}) - \phi(t,x,t^{n+1},x_j)}{\Delta_{j-1/2}}\Big\} \\
&\qquad \cdot F(u_j^n, v(t,x))\,dx\,dt\,\Delta t \\
&+ \sum_{n=0}^{N-1}\sum_{j\in\mathbb{Z}}\int_0^{t^N}\int_{\mathbb{R}} \Big\{ -\alpha_{j+1/2}\frac{\phi(t,x,t^{n+1},x_j) - \phi(t,x,t^{n+1},x_{j+1})}{\Delta_{j+1/2}} \\
&\qquad + \alpha_{j-1/2}\frac{\phi(t,x,t^{n+1},x_{j-1}) - \phi(t,x,t^{n+1},x_j)}{\Delta_{j-1/2}}\Big\} \\
&\qquad \cdot \mathcal{N}(u_j^n, v(t,x))\,dx\,dt\,\Delta t.
\end{aligned}$$

The function $U(\cdot)$ is nothing but $|\cdot|$, and $F(u,c)$ and $\mathcal{N}(u,c)$ are defined as follows:

$$F(u,c) = \int_c^u f'(s)\,U'(s-c)\,ds, \quad \mathcal{N}(u,c) = \int_c^u N'(s)\,U'(s-c)\,ds. \tag{3.1}$$

The function ϕ is given by

$$\phi(t,x,t',x_j) = \frac{1}{\Delta_j}\int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(t,x,t',s)\,ds. \tag{3.2}$$

where the function $\varphi = \varphi(t,x,t',x')$ is defined as follows:

$$\varphi = w_{\epsilon_t}(t-t')\,\eta_{\epsilon_x}(x-x'), \qquad (x,t),(x',t') \in \mathbb{R}\times\mathbb{R}^+, \tag{3.3a}$$

where ϵ_t and ϵ_x are two arbitrary positive numbers and

$$w_{\epsilon_t}(s) = \frac{1}{\epsilon_t}\,w(\frac{s}{\epsilon_t}), \quad \eta_{\epsilon_x}(s) = \frac{1}{\epsilon_x}\,\eta(\frac{s}{\epsilon_x}), \tag{3.3b}$$

for any $s \in \mathbb{R}$. For future reference, we also set

$$W(t) = \int_0^t w_{\epsilon_t}(s)\,ds. \tag{3.3c}$$

Finally, the functions w and η are smooth approximations to $\chi \equiv \chi_0$ and χ_ϵ, where

$$\chi_\epsilon(x) = \begin{cases} (1+\epsilon)/2, & \text{for } |x| \le (1-\epsilon)/(1+\epsilon), \\ (1+\epsilon)^2\,(1-|x|)/4\,\epsilon, & \text{for } |x| \in [(1-\epsilon)/(1+\epsilon), 1], \\ 0, & \text{elsewhere .} \end{cases}$$

It is easy to verify that we can find a sequence of functions η such that

$$\lim_{\eta\to\chi_\epsilon} |\,\eta\,|_{TV(\mathbb{R})} = |\,\chi_\epsilon\,|_{TV(\mathbb{R})} = 1+\epsilon, \tag{3.4a}$$

$$\lim_{\eta\to\chi_\epsilon} |\,\eta'\,|_{TV(\mathbb{R})} = |\,\chi'_\epsilon\,|_{TV(\mathbb{R})} = 2+\epsilon+1/\epsilon. \tag{3.4b}$$

$$\lim_{\eta\to\chi_\epsilon} \|\,\eta\,\|_{L^\infty(\mathbb{R})} = \|\,\chi_\epsilon\,\|_{L^\infty(\mathbb{R})} = (1+\epsilon)/2. \tag{3.4c}$$

b. Estimate of $E_{diss}(u_h, v; t^n)$. To estimate $E_{diss}(u_h, v; t^n)$, it is enough to follow the proof of the corresponding result in [4, Proposition 7.1].

Proposition 3.1. *Under condition* (2.3f) *on the viscosity term N, and if the Courant-Friedrichs-Levy* (CFL) *condition* (2.4) *is satisfied, the local rate of entropy dissipation $LRED_j^n(c)$ is nonnegative. Hence*

$$-E_{diss}(u_h, v; t^n) \le 0.$$

Sketch of the proof. The above conditions ensure that the functions $p_j^i(s)$, $i = 1, 2, 3$, are nondecreasing in s. The result follows from this fact and the definition of $E_{diss}(u_h, v; t^n)$. ☐

c. Estimate of $E^\star_{div}(u_h, v; t^N)$.

Proposition 3.2. *We have*

$$\lim_{w\to\chi} \sup_{1\le n\le N} \big\{E^\star_{div}(u, v; t^n)/W(t^n)\big\} \le TEW_{visc} + TEW_{cons} + TEW_{h.o.t.},$$

where

$$TEW_{visc}(u, v; t^n) \le \; C_0 \left\{ 2\,\frac{|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_x}\,\big(1+\frac{\Delta t}{\epsilon_t}\big)\right\} \|\nu_v\|,$$

$$TEW_{cons}(u, v; t^n) \le 2\,C_1\big(1+\frac{\Delta t}{\epsilon_t}\big)\big(1+\frac{(\Delta x)^{1/2}}{\epsilon_x}\big)\|\eta\|_{L^\infty(\mathbb{R})}\big(|\delta|_{var,1/2}+|\alpha|_{var,1/2}\big),$$

$$TEW_{h.o.t.}(u, v; t^n) \le C_1 \left\{ \frac{(\Delta t)^2\,|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_t\,\epsilon_x}\,\|\,f'\,\| + 2\,\frac{(\Delta x)^2\,|\,\eta'\,|_{TV(\mathbb{R})}}{\epsilon_x^2}\,\big(1+\frac{\Delta t}{\epsilon_t}\big)\right\},$$

where $C_0 = t^n|v_0|_{TV(\mathbb{R})}$ and $C_1 = C_0\,\|f'(v)\|$.

To prove this result, we proceed in several steps.

First step: Relating the dual form $E^\star_{div}(u, v; t)$ to the truncation error. We start by suitably relating the dual form to the truncation error. To do that, we will need the following averages of the function φ:

$$\overline{\phi}(t, x, t^{n+1}, x_j) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \phi(t, x, s, x_j)\, ds, \tag{3.5a}$$

$$\hat{\phi}(t, x, t^{n+1}, x_{j+1/2}) = \frac{1}{\Delta_{j+1/2}}\int_{-1/2}^{1/2}\int_{x_j+\Delta_j\,\rho}^{x_{j+1}+\Delta_{j+1}\,\rho} \varphi(t, x, t^{n+1}, s)\, ds\, d\rho, \tag{3.5b}$$

where $\hat{\phi}(t, x, t^{n+1}, x_{j+1/2})$ has been defined in such a way that the following equality holds:

$$\hat{\phi}_x(t, x, t^{n+1}, x_{j+1/2}) = \frac{\phi(t, x, t^{n+1}, x_j) - \phi(t, x, t^{n+1}, x_{j+1})}{\Delta_{j+1/2}}. \tag{3.5c}$$

With the above notation, we have the following upper bound for $E^\star_{div}(u_h, v; t^N)$.

Lemma 3.3. *We have*

$$\begin{aligned} E^{\star}_{div}(u_h, v; t^N) \leq & \sum_{n=0}^{N-1} \sum_{j \in \mathbb{Z}} \int_0^{t^N} \int_{\mathbb{R}} \frac{\Delta t}{2} \Delta_j \phi_{xt}(t, x, t^{n+1}, x_j)\, F(u_j^n, v(t,x))\ dx\, dt\, \Delta t \\ & + \sum_{n=0}^{N-1} \sum_{j \in \mathbb{Z}} \int_0^{t^N} \int_{\mathbb{R}} \tilde{\Psi}_j^n(t,x)\, \mathcal{N}(u_j^n, v(t,x))\, dx\, dt\, \Delta t \\ & - \sum_{n=0}^{N-1} \sum_{j \in \mathbb{Z}} \int_0^{t^N} \int_{\mathbb{R}} \tilde{\Phi}_j^n(t,x)\, F(u_j^n, v(t,x))\ dx\, dt\, \Delta t, \end{aligned}$$

where

$$\begin{aligned} \tilde{\Phi}_j^n(t,x) = & \ a_{j+1/2}\, \hat{\phi}_x(t, x, t^{n+1}, x_{j+1/2}) + b_{j-1/2}\, \hat{\phi}_x(t, x, t^{n+1}, x_{j-1/2}) \\ & - \Delta_j\, \overline{\phi}_x(t, x, t^{n+1}, x_j) + \frac{\Delta t}{2} \Delta_j \phi_{xt}(t, x, t^{n+1}, x_j), \end{aligned}$$

and

$$\tilde{\Psi}_j^n(t,x) = -\,\alpha_{j+1/2} \hat{\phi}_x(t, x, t^{n+1}, x_{j+1/2}) + \alpha_{j-1/2} \hat{\phi}_x(t, x, t^{n+1}, x_{j-1/2}).$$

To prove this result, we use the fact that v is the entropy solution and make some algebraic manipulations; see the proof of the similar result [4, Proposition 7.9].

Next, we need to relate the functions ϕ and $\hat{\phi}$ as defined in (3.5). The relations we need are displayed in the following result, which can be obtained by using simple Taylor expansions.

Lemma 3.4. *We have*

$$\begin{aligned} \hat{\phi}(t, x, t^{n+1}, x_{j+1/2}) = & \ \phi(t, x, t^{n+1}, x_j) - \frac{\Delta_j + 2\Delta_{j+1}}{6} \phi_x(t, x, t^{n+1}, x_j) \\ & + \int_{x_{j+1/2}}^{x_{j+3/2}} P_j^+(x')\, \varphi_{x'x'}(t, x, t^{n+1}, x')\, dx' \\ & + \int_{x_{j-1/2}}^{x_{j+1/2}} Q_j^+(x')\, \varphi_{x'x'}(t, x, t^{n+1}, x')\, dx', \end{aligned}$$

and

$$\begin{aligned} \hat{\phi}(t, x, t^{n+1}, x_{j-1/2}) = & \ \phi(t, x, t^{n+1}, x_j) + \frac{\Delta_j + 2\Delta_{j-1}}{6} \phi_x(t, x, t^{n+1}, x_j) \\ & + \int_{x_{j-1/2}}^{x_{j+1/2}} Q_j^-(x')\, \varphi_{x'x'}(t, x, t^{n+1}, x')\, dx' \\ & + \int_{x_{j-3/2}}^{x_{j-1/2}} P_j^-(x')\, \varphi_{x'x'}(t, x, t^{n+1}, x')\, dx'. \end{aligned}$$

The polynomials $P^{\pm}$ and $Q^{\pm}$ are given by

$$P^+(x') = \frac{(x_{j+3/2} - x')^3}{6\,\Delta_{j+1/2}\,\Delta_{j+1}}, \quad P^-(x') = \frac{(x' - x_{j-3/2})^3}{6\,\Delta_{j-1/2}\,\Delta_{j-1}},$$

$$Q^+(x') = \frac{(x' - x_{j-1/2} - \Delta_{j+1/2})^3 + (\Delta_{j+1/2})^3}{6\,\Delta_{j+1/2}\,\Delta_j} + \frac{\Delta_{j+1} - \Delta_j}{12\,\Delta_j}(x' - x_{j-1/2}),$$

$$Q^-(x') = \frac{(x_{j+1/2} - x' - \Delta_{j-1/2})^3 + (\Delta_{j-1/2})^3}{6\,\Delta_{j-1/2}\,\Delta_j} + \frac{\Delta_{j-1} - \Delta_j}{12\,\Delta_j}(x_{j+1/2} - x').$$

With the above lemma, we can now rewrite the upper bound of $E^\star_{div}(u_h, v; t^N)$ as the sum of three terms. The first, $TE_{visc}(u, v; t^N)$, is that part of the truncation error which contains the information of the viscosity of the numerical scheme. The second term, $TE_{cons}(u, v; t^N)$, contains information concerning the *consistency* of the numerical scheme; indeed, if the scheme is consistent, then $TE_{cons}(u, v; t^N) = 0$. We emphasize that to define this term, the definition of $\delta_{j+1/2}$, (2.5), must be used. The third term, $TE_{h.o.t.}(u, v; t^N)$, contains the high-order terms in the truncation error and, as expected, will be dominated by the term $TE_{visc}(u, v; t^N)$ and $TE_{cons}(u, v; t^N)$.

Lemma 3.5. *We have*

$$E^\star_{div}(u_h, v; t^N) \le TE_{visc}(u, v; t^N) + TE_{cons}(u, v; t^N) + TE_{h.o.t.}(u, v; t^N),$$

where

$$\begin{aligned}
TE_{visc}(u, v; t^N) = \; & \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \int_0^{t^N} \int_{\mathbb{R}} VISC^n_j(v(t,x); t, x)\; dx\, dt\, \Delta t \\
& - \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} F(u^n_j, v(t^N, x)) \frac{\Delta t}{2} \phi_x(t^N, x, t^{n+1}, x_j)\, dx\, \Delta_j\, \Delta t \\
& + \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} F(u^n_j, v(t^0, x)) \frac{\Delta t}{2} \phi_x(0, x, t^{n+1}, x_j)\, dx\, \Delta_j\, \Delta t, \\
TE_{cons}(u, v; t^N) = \; & \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \int_0^{t^N} \int_{\mathbb{R}} CONS^n_j(v(t,x); t, x)\; dx\, dt\, \Delta t, \\
TE_{h.o.t.}(u, v; t^N) = & \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \int_0^{t^N} \int_{\mathbb{R}} HOT^n_j(v(t,x); t, x)\; dx\, dt\, \Delta t \\
& + \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} F(u^n_j, v(t^N, x)) \frac{\Delta t}{2} \phi_x(t^N, x, t^{n+1}, x_j)\, dx\, \Delta_j\, \Delta t \\
& - \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} F(u^n_j, v(t^0, x)) \frac{\Delta t}{2} \phi_x(0, x, t^{n+1}, x_j)\, dx\, \Delta_j\, \Delta t.
\end{aligned}$$

The 'viscosity' term $VISC^n_j(c; t, x)$ is given by

$$VISC^n_j(c; t, x) = \overset{time}{VISC}{}^n_j(c; t, x) + \overset{cent}{VISC}{}^n_j(c; t, x) + \overset{visc}{VISC}{}^n_j(c; t, x),$$

where

$$\overset{time}{VISC}{}_j^n(c;t,x) = F(u_j^n,c)\,\Big\{\frac{\Delta t}{2}\Delta_j\,\phi_{xt}(t,x,t^{n+1},x_j)\Big\},$$

$$\overset{cent}{VISC}{}_j^n(c;t,x) = F(u_j^n,c)\left\{\, a_{j+1/2}\frac{\Delta_j + 2\,\Delta_{j+1}}{6} - b_{j-1/2}\frac{\Delta_j + 2\,\Delta_{j-1}}{6}\right\}\phi_{xx}(t,x,t^{n+1},x_j),$$

$$\overset{visc}{VISC}{}_j^n(c;t,x) = \mathcal{N}(u_j^n,c)\left\{\, \alpha_{j+1/2}\,\frac{\Delta_j + 2\Delta_{j+1}}{6} + \alpha_{j-1/2}\,\frac{\Delta_j + 2\Delta_{j-1}}{6}\right\}\phi_{xx}(t,x,t^{n+1},x_j).$$

The 'consistency' term $CONS_j^n(c;t,x)$ *is given by*

$$CONS_j^n(c;t,x) = \overset{cent}{CONS}{}_j^n(c;t,x) + \overset{visc}{CONS}{}_j^n(c;t,x),$$

where

$$\overset{cent}{CONS}{}_j^n(c;t,x) = F(u_j^n,c)\,(-\delta_{j+1/2} + \delta_{j-1/2})\,\phi_x(t,x,t^{n+1},x_j),$$

$$\overset{visc}{CONS}{}_j^n(c;t,x) = \mathcal{N}(u_j^n,c)\,(-\alpha_{j+1/2} + \alpha_{j-1/2})\,\phi_x(t,x,t^{n+1},x_j).$$

Finally, the 'high-order' term $HOT_j^n(c;t,x)$ *is given by*

$$HOT_j^n(c;t,x) = \overset{time}{HOT}{}_j^n(c;t,x) + \overset{cent}{HOT}{}_j^n(c;t,x) + \overset{visc}{HOT}{}_j^n(c;t,x),$$

where

$$\overset{time}{HOT}{}_j^n(c;t,x) = F(u_j^n,c)\,\Big\{\Delta_j\,\overline{\phi}_x(t,x,t^{n+1},x_j) - \Delta_j\,\phi_x(t,x,t^{n+1},x_j) + \frac{\Delta t}{2}\Delta_j\,\phi_{xt}(t,x,t^{n+1},x_j)\Big\},$$

$$\begin{aligned}\overset{cent}{HOT}{}_j^n(c;t,x) = \Big\{ &- a_{j+1/2}\int_{x_{j+1/2}}^{x_{j+3/2}} P_j^+(x')\,\varphi_{x'x'x}(t,x,t^{n+1},x')\,dx' \\ &- a_{j+1/2}\int_{x_{j-1/2}}^{x_{j+1/2}} Q_j^+(x')\,\varphi_{x'x'x}(t,x,t^{n+1},x')\,dx' \\ &- b_{j-1/2}\int_{x_{j-1/2}}^{x_{j+1/2}} Q_j^-(x')\,\varphi_{x'x'x}(t,x,t^{n+1},x')\,dx' \\ &- b_{j-1/2}\int_{x_{j-3/2}}^{x_{j-1/2}} P_j^-(x')\,\varphi_{x'x'x}(t,x,t^{n+1},x')\,dx'\Big\}\,F(u_j^n,c),\end{aligned}$$

$$\begin{aligned}\overset{visc}{HOT}{}_j^n(c;t,x) = \Big\{ &- \alpha_{j+1/2}\int_{x_{j+1/2}}^{x_{j+3/2}} P_j^+(x')\,\varphi_{x'x'x}(t,x,t^{n+1},x')\,dx' \\ &- \alpha_{j+1/2}\int_{x_{j-1/2}}^{x_{j+1/2}} Q_j^+(x')\,\varphi_{x'x'x}(t,x,t^{n+1},x')\,dx' \\ &+ \alpha_{j-1/2}\int_{x_{j-1/2}}^{x_{j+1/2}} Q_j^-(x')\,\varphi_{x'x'x}(t,x,t^{n+1},x')\,dx' \\ &+ \alpha_{j-1/2}\int_{x_{j-3/2}}^{x_{j-1/2}} P_j^-(x')\,\varphi_{x'x'x}(t,x,t^{n+1},x')\,dx'\Big\}\,\mathcal{N}(u_j^n,c).\end{aligned}$$

Second step: Estimating $TE_{visc}(u,v;t^N)$. In this section, we prove the following result.

Lemma 3.6. *If* (2.3) *and* (2.4) *hold, we have*

$$TE_{visc}(u,v;t^N) \,\le\, 2\, C_0 \,\frac{|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_x}\,\big(1+\frac{\Delta t}{\epsilon_t}\big)\,\|\nu_v\|\,\Delta x,$$

where $C_0 = \, t^N |v_0|_{TV(\mathbb{R})}\, W(t^N)$.

We need the following auxiliary result whose proof can be found in the proof of [4, Lemma 7.13].

Lemma 3.7. *We have*

$$\sup_{t\in(0,t^N)} \left\{ \sum_{n=0}^{N-1} w_{\epsilon_t}(t-t^{n+1})\,\Delta t \right\} \le\, 2\,\big(1+\frac{\Delta t}{\epsilon_t}\big)\, W(t^N).$$

Proof of Lemma 3.6 . As in the proof of [4, Lemma 7.11], it is enough to consider an entropy solution v that is everywhere smooth except on a single discontinuity curve $\mathcal{C} = \{(x(t),t) : t\in(0,t^N)\}$.

We have

$$TE_{visc}(u,v;t^N) = \sum_{n=0}^{N-1}\sum_{j\in\mathbb{Z}} \Xi(t^{n+1},x_j)\,\Delta_j\Delta t,$$

where

$$\begin{aligned}
\Xi(t^{n+1},x_j) = \int_0^{t^N}\!\!\int_{\mathbb{R}} \Bigg\{ & F(u_j^n, v(t,x))\frac{\Delta t}{2}\phi_{xt}(t,x,t^{n+1},x_j) \\
&+ F(u_j^n, v(t,x))\Bigg(a_{j+1/2}\frac{\Delta_j+2\Delta_{j+1}}{6\Delta_j} \\
&\qquad\qquad - b_{j-1/2}\frac{\Delta_j+2\Delta_{j-1}}{6\Delta_j}\Bigg)\phi_{xx}(t,x,t^{n+1},x_j) \\
&+ \mathcal{N}(u_j^n, v(t,x))\Bigg(\alpha_{j+1/2}\frac{\Delta_j+2\Delta_{j+1}}{6\Delta_j} \\
&\qquad\qquad + \alpha_{j-1/2}\frac{\Delta_j+2\Delta_{j-1}}{6\Delta_j}\Bigg)\phi_{xx}(t,x,t^{n+1},x_j)\Bigg\}\,dxdt \\
&- \int_{\mathbb{R}} F(u_j^n, v(t^N,x))\frac{\Delta t}{2}\phi_x(t^N,x,t^{n+1},x_j)\,dx \\
&+ \int_{\mathbb{R}} F(u_j^n, v(0,x))\frac{\Delta t}{2}\phi_x(0,x,t^{n+1},x_j)\,dx.
\end{aligned}$$

A couple of integrations by parts yields

$$\begin{aligned}\Xi(t^{n+1},x_j) = &-\int_0^{t^N}\int_{-\infty}^{x(t)}\left\{\frac{1}{2}\Delta t F_t + \left(a_{j+1/2}\frac{\Delta_j+2\Delta_{j+1}}{6\Delta_j} - b_{j-1/2}\frac{\Delta_j+2\Delta_{j-1}}{6\Delta_j}\right)F_x\right.\\ &\left.+\left(\alpha_{j+1/2}\frac{\Delta_j+2\Delta_{j+1}}{6\Delta_j}+\alpha_{j-1/2}\frac{\Delta_j+2\Delta_{j-1}}{6\Delta_j}\right)\mathcal{N}_x\right\}\phi_x\,dxdt\\ &-\int_0^{t^N}\int_{x(t)}^{\infty}\left\{\frac{1}{2}\Delta t F_t + \left(a_{j+1/2}\frac{\Delta_j+2\Delta_{j+1}}{6\Delta_j} - b_{j-1/2}\frac{\Delta_j+2\Delta_{j-1}}{6\Delta_j}\right)F_x\right.\\ &\left.+\left(\alpha_{j+1/2}\frac{\Delta_j+2\Delta_{j+1}}{6\Delta_j}+\alpha_{j-1/2}\frac{\Delta_j+2\Delta_{j-1}}{6\Delta_j}\right)\mathcal{N}_x\right\}\phi_x\,dxdt\\ &-\int_0^{t^N}\left\{-\frac{1}{2}\Delta t[F]\frac{[f]}{[v]} + \left(a_{j+1/2}\frac{\Delta_j+2\Delta_{j+1}}{6\Delta_j} - b_{j-1/2}\frac{\Delta_j+2\Delta_{j-1}}{6\Delta_j}\right)[F]\right.\\ &\left.+\left(\alpha_{j+1/2}\frac{\Delta_j+2\Delta_{j+1}}{6\Delta_j}+\alpha_{j-1/2}\frac{\Delta_j+2\Delta_{j-1}}{6\Delta_j}\right)[\mathcal{N}]\right\}\phi_x\,dt,\end{aligned}$$

where the last integral is understood to be along the discontinuity line $\mathcal{C}$.

The jump of a function $G(v)$ across $\mathcal{C}$ at a point $(x(t),t)$ is denoted $[G] = G(v(t,x(t)+0)) - G(v(t,x(t)-0))$.

Setting $\tilde{\nu}_j = \nu_j(u; v(t,x(t)-0), v(t,x(t)+0))$, where $\nu_j(u;v^-,v^+)$ is defined in Theorem 2.1, we get

$$\Xi(t^{n+1},x_j) = -\int_0^{t^N}\left\{\int_{-\infty}^{x(t)}\tilde{\nu}_j v_x\phi_x\,dx + \tilde{\nu}_j[v]\phi_x + \int_{x(t)}^{\infty}\tilde{\nu}_j v_x\phi_x\,dx\right\}dt.$$

Proceeding again as in [4], we set

$$\|\tilde{\nu}_j\| = \sup_{\substack{t\in(0,t^N)\\ x\in\mathbb{R}}}\sup_{u\in\mathbb{R}}|\tilde{\nu}_j|,$$

and thus

(3.6)
$$|\Xi(t^{n+1},x_j)| \le \|\tilde{\nu}_j\|\int_0^{t^N}\left\{\int_{-\infty}^{x(t)}|v_x||\phi_x|\,dx + |[v]||\phi_x| + \int_{x(t)}^{\infty}|v_x||\phi_x|\,dx\right\}dt.$$

We would like to analyze further the dependence of $\tilde{\nu}_j$ with respect to u. Setting $\nu^{\#}(u) = \nu_j(u;v^-,v^+)$ and taking into account (3.1), direct calculations yield

$$\begin{aligned}\partial_u\nu^{\#}(u) = \frac{1}{2}\frac{U'(u-v^+)-U'(u-v^-)}{v^+-v^-}&\left\{\left(\alpha_{j+1/2}\frac{\Delta_j+2\Delta_{j+1}}{3\Delta_j}\right.\right.\\ &\left.+\alpha_{j-1/2}\frac{\Delta_j+2\Delta_{j-1}}{3\Delta_j}\right)N'(u)\\ +\left(a_{j+1/2}\frac{\Delta_j+2\Delta_{j+1}}{3\Delta_j} - b_{j-1/2}\frac{\Delta_j+2\Delta_{j-1}}{3\Delta_j}\right)&f'(u)\\ &\left.-\Delta t\frac{f(v^+)-f(v^-)}{v^+-v^-}f'(u)\right\}.\end{aligned}$$

Therefore $\nu^{\#}(u)$ is constant for any value of u which does not lie between v^- and v^+. This implies that

$$\sup_{u\in\mathbb{R}} |\nu^{\#}(u)| = \sup_{u\in[v^-\wedge v^+, v^-\vee v^+]} |\nu^{\#}(u)| \equiv \nu_j(v^-, v^+)\,\Delta x.$$

Inserting $\nu_j(v^-, v^+)$ in the bound (3.6) and using the definition of $\|\nu_v\|$, we get, since v is the entropy solution

$$\begin{aligned} TE_{visc}(u, v; t^N) &\le T_{aux}\|\nu_v\|\,\Delta x \int_0^{t^N} \left\{ \int_{-\infty}^{x(t)} |v_x|\,dx + |[v]| + \int_{x(t)}^{\infty} |v_x| dx \right\} dt \\ &\le T_{aux} t^N |v|_{L^\infty(0,t^N;TV(\mathbb{R}))} \|\nu_v\|\,\Delta x \\ &\le T_{aux} t^N |\, v_0\,|_{TV(\mathbb{R})} \|\nu_v\|\,\Delta x, \end{aligned}$$

where

$$\begin{aligned} T_{aux} &= \sup_{\substack{t\in(0,t^n)\\ x\in\mathbb{R}}} \left\{ \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} |\phi_x(t, x, t^{n+1}, x_j)|\,\Delta_j \Delta t \right\} \\ &\le \left\{ \int_{\mathbb{R}} \frac{1}{\epsilon_x} |\,\eta'(y)\,|\,dy \right\} \sup_{t\in(0,t^N)} \left\{ \sum_{n=0}^{N-1} w_{\epsilon_t}(t - t^{n+1})\,\Delta t \right\}. \end{aligned}$$

Taking Lemma 3.7 into account achieves the proof. □

Third step: Estimating $TE_{cons}(u, v; t^N)$.

Lemma 3.8. *We have*

$$TE_{cons}(u, v; t^N) \le 2\, C_1 \big(1 + \frac{\Delta t}{\epsilon_t}\big) \big(1 + \frac{(\Delta x)^{1/2}}{\epsilon_x}\big) \,\|\,\eta\,\|_{L^\infty(\mathbb{R})} \big(|\,\delta\,|_{var,1/2} + |\,\alpha\,|_{var,1/2}\big),$$

where $C_1 = t^N |v_0|_{TV(\mathbb{R})}\, \|N'(v)\|\, W(t^N)$.

Note that if the scheme is consistent, the upper bound for $TE_{cons}(u, v; t^N)$ is equal to zero, as expected.

We will need the following simple auxiliary result.

Lemma 3.9. *We have*

$$\frac{1}{\epsilon_x} \sum_{|\,x - x_j| \le \epsilon_x} |\,\delta_{j+1/2} - \delta_{j-1/2}\,| \le \big(\,1 + \frac{(\Delta x)^{1/2}}{\epsilon_x}\,\big)\,|\,\delta\,|_{var,1/2}.$$

Proof of Lemma 3.8. The consistency error $TE_{cons}(u, v; t^N)$ is the sum of two terms of the form

$$\Theta = -\sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \int_0^{t^N} \int_{\mathbb{R}} \big(G(u_j^n, v(t, x))\big)\, \phi(t, x, t^{n+1}, x_j)_x\, (\zeta_{j+1/2} - \zeta_{j-1/2})\, dx\, dt\, \Delta t,$$

one of which has $\zeta = \delta$ and $G = F$, and the other $\zeta = \alpha$ and $G = \mathcal{N}$. Thus, it is enough to get an estimate for Θ.

To do that, we assume that the entropy solution v is smooth; see the proof [4, Proposition 5.5]. First, we integrate by parts in the x variable to obtain

$$\begin{aligned} \Theta &= \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \int_0^{t^N} \int_{\mathbb{R}} \big(G(u_j^n, v(t, x))\big)_x\, \phi(t, x, t^{n+1}, x_j)\, (\zeta_{j+1/2} - \zeta_{j-1/2})\, dx\, dt\, \Delta t \\ &\le T_{aux}\, t^N \,\|\, N'(v)\,\|\, |v_0|_{TV(\mathbb{R})}, \qquad \text{by (2.3f) and (3.1)}, \end{aligned}$$

where

$$T_{aux} = \sup_{\substack{t\in(0,t^n)\\ x\in\mathbb{R}}} \left\{ \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \phi(t,x,t^{n+1},x_j) \,|\, \zeta_{j+1/2} - \zeta_{j-1/2} \,|\, \Delta_j \Delta t \right\}$$

$$\leq \|\,\eta\,\|_{L^\infty(\mathbb{R})} \sup_{t\in(0,t^N)} \left\{ \sum_{n=0}^{N-1} w_{\epsilon_t}(t-t^{n+1})\Delta t \right\} \left\{ \frac{\sum_{|x-x_j|\leq\epsilon_x} |\zeta_{j+1/2} - \zeta_{j-1/2}|}{\epsilon_x} \right\}.$$

Taking Lemmas 3.7 and 3.9 into account achieves the proof. □

Fourth step: Estimating $TE_{h.o.t.}(u,v;t^N)$.

Lemma 3.10. *Suppose that the conditions* (2.3) *are satisfied. Then,*

$$TE_{h.o.t.}(u,v;t^N) \leq C_1 \left\{ \frac{(\Delta t)^2 \,|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_t\,\epsilon_x} \,\|\, f' \,\|\, + 2\, \frac{(\Delta x)^2 \,|\,\eta'\,|_{TV(\mathbb{R})}}{\epsilon_x^2} \left(1 + \frac{\Delta t}{\epsilon_t}\right) \right\},$$

where $C_1 = t^n |v_0|_{TV(\mathbb{R})} \|f'(v)\|\, W(t^n)$.

To prove the above result, we rewrite $TE_{h.o.t.}(u,v;t^N)$ as the sum

$$\overset{time}{TE}_{h.o.t.}(u,v;t^N) + \overset{cent}{TE}_{h.o.t.}(u,v;t^N) + \overset{visc}{TE}_{h.o.t.}(u,v;t^N),$$

with the obvious notation, and estimate each of the above three terms. The following estimate can be easily obtained by following the techniques used in the proofs of [4, Lemma 7.12] and [4, Lemma 7.13] and by using (2.3f).

Lemma 3.11. *We have*

$$\overset{time}{TE}_{h.o.t.}(u,v;t^N) \leq C_1 \,\frac{(\Delta t)^2 \,|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_t\,\epsilon_x} \,\|\, f' \,\|.$$

To estimate the two remaining terms, we need a couple of simple auxiliary results.

Lemma 3.12. *We have*

$$\begin{aligned}
p_j^+ &\equiv \|\, P_j^+ \,\|_{L^\infty(x_{j+1/2},x_{j+3/2})} \leq \frac{1}{3}\,\Delta_{j+1},\\
q_j^+ &\equiv \|\, Q_j^+ \,\|_{L^\infty(x_{j+1/2},x_{j+3/2})} \leq \frac{1}{6}\,\max\{\Delta_j,\Delta_{j+1}\},\\
q_j^- &\equiv \|\, Q_j^- \,\|_{L^\infty(x_{j+1/2},x_{j+3/2})} \leq \frac{1}{6}\,\max\{\Delta_{j-1},\Delta_j\},\\
p_j^- &\equiv \|\, P_j^- \,\|_{L^\infty(x_{j+1/2},x_{j+3/2})} \leq \frac{1}{3}\,\Delta_{j-1}.
\end{aligned}$$

This result follows easily from the definition of the polynomials $P_j^\pm$ and $Q_j^\pm$ in Lemma 3.4.

Lemma 3.13. *Suppose the conditions* (2.3) *are satisfied. Then we have*

$$\kappa_j \equiv |\,a_{j-1/2}\,|\,p_{j-1}^+ + |\,a_{j+1/2}\,|\,q_j^+ + |\,b_{j-1/2}\,|\,q_j^- + |\,b_{j+1/2}\,|\,p_{j+1}^- \leq \frac{1}{2}\,(\,\Delta x\,)^2,$$

$$\tilde\kappa_j \equiv |\,\alpha_{j-1/2}\,|\,p_{j-1}^+ + |\,\alpha_{j+1/2}\,|\,q_j^+ + |\,\alpha_{j-1/2}\,|\,q_j^- + |\,\alpha_{j+1/2}\,|\,p_{j+1}^- \leq \frac{1}{2}\,(\,\Delta x\,)^2.$$

The proof follows easily from the preceding lemma and conditions (2.3e) and (2.3f). We can now estimate the two remaining terms.

Lemma 3.14. *Suppose that the conditions* (2.3) *are satisfied. Then,*

$$\overset{cent}{TE}_{h.o.t.}(u,v;t^N)/W(t^N) \leq C_1 \frac{(\Delta x)^2 \,|\, \eta' \,|_{TV(\mathbb{R})}}{\epsilon_x^2} \big(1+\frac{\Delta t}{\epsilon_t}\big),$$
$$\overset{visc}{TE}_{h.o.t.}(u,v;t^N)/W(t^N) \leq C_1 \frac{(\Delta x)^2 \,|\, \eta' \,|_{TV(\mathbb{R})}}{\epsilon_x^2} \big(1+\frac{\Delta t}{\epsilon_t}\big).$$

Proof. We only have to prove the first estimate since the second is similar and the upper bounds for κ_j and $\tilde{\kappa}_j$ given in Lemma 3.13 are identical. To do that, let us rewrite $\overset{cent}{TE}_{h.o.t.}(u,v;t^N)$ as the sum $\Theta_1+\Theta_2+\Theta_3+\Theta_4$ with obvious notation. Next, let us estimate the first term

$$\Theta_1 = \sum_{n=0}^{N-1}\sum_{j\in\mathbb{Z}}\int_0^{t^N}\int_{\mathbb{R}} F(u_j^n, v(t,x))\, a_{j+1/2} \cdot \int_{x_{j+1/2}}^{x_{j+3/2}} P_j^+(x')\,\varphi_{x'x'x}(t,x,t^{n+1},x')\,dx'\;dx\,dt\,\Delta t.$$

We can assume that the entropy solution v is smooth since the general case can be obtained by a standard density argument; see the proof [4, Proposition 5.5]. Integrating by parts in the variable x, taking absolute values, and changing the index j, we get

$$\Theta_1 \leq \int_0^{t^N}\int_{\mathbb{R}} |\, f'(v(t,x))\,|\,|\, v_x(t,x)\,|\, \mathcal{T}_1(t,x)\,dx\,dt,$$

where

$$\mathcal{T}_1(t,x) = \sum_{n=0}^{N-1}\sum_{j\in\mathbb{Z}} |\, a_{j-1/2}\,|\; p_{j-1}^+ \int_{x_{j-1/2}}^{x_{j+1/2}} |\,\varphi_{x'x'}(t,x,t^{n+1},x')\,|\,dx'\,\Delta t.$$

Proceeding in a similar way with Θ_2, Θ_3, and Θ_4, we get

$$\overset{cent}{TE}_{h.o.t.}(u,v;t^N) \leq \int_0^{t^N}\int_{\mathbb{R}} |\, f'(v(t,x))\,|\,|\, v_x(t,x)\,|\, \mathcal{T}(t,x)\,dx\,dt$$

where

$$\mathcal{T}(t,x) = \sum_{n=0}^{N-1}\sum_{j\in\mathbb{Z}} \kappa_j \int_{x_{j-1/2}}^{x_{j+1/2}} |\,\varphi_{x'x'}(t,x,t^{n+1},x')\,|\,dx'\,\Delta t.$$

Thus, by (2.3f),

$$\overset{cent}{TE}_{h.o.t.}(u,v;t^N) \leq T_{aux}\, t^N \,\|\, N' \,\|\, |v_0\,|_{TV(\mathbb{R})},$$

where

$$\begin{aligned} T_{aux} &= \sup_{x\in\mathbb{R}, t\in(0,t^N)} \mathcal{T}(t,x) \\ &\leq \frac{(\Delta x)^2}{2\epsilon_x^2}\left\{\int_{\mathbb{R}} \eta''(y)\,|\,dy\right\} \sup_{t\in(0,t^N)}\left\{\sum_{n=0}^{N-1} w_{\epsilon_t}(t-t^{n+1})\,\Delta t\right\} \\ &\leq \frac{(\Delta x)^2\,|\,\eta'\,|_{TV(\mathbb{R})}}{\epsilon_x^2}\big(1+\frac{\Delta t}{\epsilon_t}\big)\, W(t^N), \end{aligned}$$

by the definition of φ (3.3), by Lemma 3.13, and by Lemma 3.7. This completes the proof. □

Lemma 3.10 follows easily from Lemmas 3.11 and 3.14.

d. Proof of the error estimate. To obtain the error estimate, we proceed exactly as in [4]. If we insert the estimates obtained in §3.b and §3.c into the approximation inequality of §3.a, and we take the auxiliary function η as in (3.4), we obtain

$$\begin{aligned} e(t^N) \leq\ & 2\,e(0) + 8\,\big(\epsilon_x + \epsilon_t \| f'(v) \|\big)\,|\, v_0\,|_{TV(\mathbb{R})} + 2\,\|\, f'(v)\,\|\,|\, v_0\,|_{TV(\mathbb{R})}\,\Delta t \\ & + 2\,C_0 \left\{ 2\,\frac{|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_x}\,\big(1 + \frac{\Delta t}{\epsilon_t}\big) \right\} \|\nu_v\| \\ & + 4\,C_1\,\big(1 + \frac{\Delta t}{\epsilon_t}\big)\,\big(1 + \frac{(\Delta x)^{1/2}}{\epsilon_x}\big)\,\|\,\eta\,\|_{L^\infty(\mathbb{R})}\,\big(|\,\delta\,|_{var,1/2} + |\,\alpha\,|_{var,1/2}\big) \\ & + 2\,C_1 \left\{ \frac{(\Delta t)^2\,|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_t\,\epsilon_x}\,\|\, f'\,\| + 2\,\frac{(\Delta x)^2\,|\,\eta'\,|_{TV(\mathbb{R})}}{\epsilon_x^2}\,\big(1 + \frac{\Delta t}{\epsilon_t}\big) \right\}, \end{aligned}$$

where $C_0 = t^N |v_0|_{TV(\mathbb{R})}$ and $C_1 = C_0\,\|\, N'(v)\,\|$. The estimate of Theorem 2.1 is then obtained by eliminating the parameter Δt by taking into account the CFL condition (2.4) and then taking the very same optimal values taken for the case treated in [4], namely,

$$\epsilon_x^\star = \sqrt{t^N\,\|\,\nu_v\,\|\,\Delta x/2}, \quad \epsilon_t = A_t\,(\Delta x)^{3/4}, \quad \epsilon = A\,(\Delta x)^{1/4}.$$

This concludes the proof of Theorem 2.1.

4. Concluding remarks

In [4], we proposed a general theory of *a priori* error estimates for scalar conservation laws, based on the original Kuznetsov approximation theory [12]. In the present paper, this approach is applied to flux-splitting monotone schemes on (Cartesian products of) nonuniform grids. The nonuniformity of the grids brings up a problem of consistency and *supraconvergence* that has no counterpart in the case of uniform grids. Indeed, the global error of these schemes seems to be insensitive to the deterioration of the part of the (formal) truncation error due to the lack of consistency of the schemes.

This *supraconvergence* phenomenon has remained unexplained until now. In this paper, we identify the proper truncation error and show that optimal error estimates can be proven without using any regularity property of the approximate solution *provided* the schemes are "consistent enough." On the other hand, we show that the regularity properties of the numerical approximation can compensate the lack of consistency of the scheme because of the special *structure* of the part of the truncation error generated by the lack of consistency of the scheme. This special structure does not have anything to do with the nonlinear nature of the problem. Instead, it is a reflection of the *consistency* of the numerical flux and the fact that the scheme is written in *conservation form.* It is thanks to this that the *supraconvergence* phenomenon takes place. Let us point out that our analysis does not rule out the possibility of *supraconvergence* for schemes written in *non-conservative* form. To settle this question, the tools provided in this paper can be easily used.

The application of our approach to problems defined on general multidimensional grids, to nonsplitting numerical fluxes, and to high-order accurate methods are the subject of forthcoming publications.

Acknowledgements

The authors want to thank John Lowengrub, Xiangrong Yang, and an anonymous referee for their comments which led to a better presentation of this paper.

References

1. M.J. Berger, R.J. Leveque, and L.G. Stern, *Finite volume methods for irregular one-dimensional grids*, Mathematics of Computation 1943–1993: a half century of computational mathematics (Vancouver, BC, 1993), Proc. Sympo. Appl. Math. 48 , Amer. Math. Soc., 1994, pp. 255–259. CMP 95:07
2. B. Cockburn, F. Coquel, and P. LeFloch, *An error estimate for finite volume methods for conservations laws*, Math. Comp. **63** (1994), 77-103. MR **95d:**65078
3. B. Cockburn and P.-A. Gremaud, *An error estimate for finite element methods for conservations laws*, SIAM J. Numer. Anal. **33** (1996), 522–554. CMP 96:12
4. ———, *A priori error estimates for numerical methods for scalar conservation laws. Part I: The general approach*, Math. Comp. **65** (1996), 533-573. MR **96g:**65089
5. B. Engquist and S. Osher, *One sided difference approximations for nonlinear conservation laws*, Math. Comp. **36** (1981), 321–351. MR **82c:**65056
6. B. García-Archilla, *A supraconvergent scheme for the Korteweg-de-Vries equation*, Numer. Math. **61** (1992), 291–310. MR **92k:**65146
7. B. García-Archilla and J.M. Sanz-Serna, *A finite difference formula for the discretization of d^3/dx^3 on nonuniform grids*, Math. Comp. **57** (1991), 239–257. MR **91j:**65143
8. A. Harten, B. Engquist, S. Osher and S. Chakravarthy, *Uniformly high order accurate essentially non-oscillatory schemes, III*, J. Comput. Phys. **71** (1987), 231-303. MR **90a:**65199
9. J.D. Hoffman, *Relationship between the truncation error of centered finite difference approximations on uniform and nonuniform meshes*, J. Comput. Phys. **46** (1982), 469–474. MR **83j:**65031
10. H.-O. Kreiss, T.A. Manteuffel, B. Swartz, B. Wendroff, and A.B. White, Jr., *Supra-convergent schemes on Irregular grids*, Math. Comp. **47** (1986), 537–554. MR **88b:**65082
11. S.N. Kružkov, *First order quasilinear equations in several independent variables*, Math. USSR Sbornik **10** (1970), 217–243. MR **42:**2159
12. N.N. Kuznetsov, *Accuracy of some approximate methods for computing the weak solutions of a first-order quasi-linear equation*, USSR Comp. Math. and Math. Phys. **16** (1976), 105–119. MR **58:**3510
13. B. J. Lucier, *A stable adaptive scheme for hyperbolic conservation laws*, SIAM J. Numer. Anal. **22** (1985), 180-203. MR **86d:**65123
14. ———, *Error bounds for the methods of Glimm, Godunov and LeVeque*, SIAM J. Numer. Anal. **22** (1985), 1074-1081. MR **88a:**65104
15. ———, *On nonlocal monotone difference schemes for scalar conservation laws*, Math. Comp. **47** (1986), 19-36. MR **87j:**65110
16. ———, *A moving mesh numerical method for hyperbolic conservation laws*, Math. Comp. **46** (1986), 59-69. MR **87m:**65141
17. T.A. Manteuffel and A.B. White, Jr., *The numerical solution of second-order boundary value problems on nonuniform meshes*, Math. Comp. **47** (1986), 511–535. MR **87m:**65116
18. H. Nessyahu and E. Tadmor, *The convergence rate of approximate solutions for nonlinear scalar conservation laws*, SIAM J. Numer. Anal. **29** (1992), 1505–1519. MR **93j:**65139
19. H. Nessyahu, E. Tadmor and T. Tassa, *The convergence rate of Godunov type schemes*, SIAM J. Numer. Anal. **31** (1994), 1–16. MR **94m:**65140
20. S. Noelle, *A note on entropy inequalities and error estimates for higher-order accurate finite volume schemes on irregular grids*, Math. Comp. **65** (1996), 1155–1163. MR **96j:**65089
21. J. Pike, *Grid adaptive algorithms for the solution of the Euler equations on irregular grids*, J. Comput. Phys. **71** (1987), 194–223. MR **88g:**65083

22. R. Sanders, *On Convergence of monotone finite difference schemes with variable spatial differencing*, Math. Comp. **40** (1983), 91–106. MR **84a:**65075
23. G. Strang, *Accurate partial difference methods II: non-linear problems*, Numer. Math. **6** (1964), 37–46. MR **29:**4215
24. E. Tadmor, *Local error estimates for discontinuous solutions of nonlinear hyperbolic equations*, SIAM J. Numer. Anal. **28** (1991), 891–906. MR **92d:**35190
25. E. Turkel, *Accuracy of schemes with nonuniform meshes for compressible fluid flow*, App. Numer. Math. **2** (1986), 529–550. MR **88m:**65137
26. J.-P. Vila, *Convergence and error estimates in finite volume schemes for general multidimensional scalar conservation laws*, Model. Math. Anal. Numer. **28** (1994), 267–295. MR **96d:**65150
27. B. Wendroff and A.B. White, Jr., *Some supraconvergent schemes for hyperbolic equations on irregular grids*, Second International Conference on Hyperbolic Problems, Aachen (1988), 671–677. CMP 21:10
28. B. Wendroff and A.B. White, Jr., *A supraconvergent scheme for nonlinear hyperbolic systems*, Comput. Math. Appl. **18** (1989), 761–767. MR **90g:**65121

School of Mathematics, University of Minnesota, 127 Vincent Hall, Minneapolis, Minnesota 55455

Center for Research in Scientific Computation and Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695-8 205