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ON A GENERALIZED PUNCTURED NEIGHBORHOOD THEOREM IN $\mathscr{L}(X)$

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ABSTRACT. Suppose that T is a bounded linear operator on a complex Banach space X. If $T^2(X)$ is closed, $T(X) \cap N(T)$ is finite dimensional, and S is a bounded linear operator on X such that S is invertible, commutes with T, and has sufficiently small norm, then $T - S$ is upper semi-Fredholm.

Throughout this paper X will denote a complex Banach space. We write $\mathscr{L}(X)$ for the set of all bounded linear operators on X. For $T \in \mathscr{L}(X)$, we denote by $N(T)$ the kernel and by $T(X)$ the range of T. The operator T is called *upper semi-Fredholm* if $T(X)$ is closed and $\dim N(T) < \infty$. We write $\sigma(T)$ for the spectrum of T. It is well known that the resolvent $R_\lambda(T) = (\lambda I - T)^{-1}$ is a holomorphic function of λ for points λ in the resolvent set $\mathbb{C} \setminus \sigma(T)$.

The aim of this paper is the following generalization of the "punctured neighborhood theorem" for upper semi-Fredholm operators:

Theorem 1. *Suppose that $T \in \mathscr{L}(X)$, T^2 has closed range, and $T(X) \cap N(T)$ is finite dimensional. Then*:

(a) *$T - S$ is upper semi-Fredholm whenever $S \in \mathscr{L}(X)$ is invertible, $TS = ST$, and $\|S\|$ is sufficiently small. Furthermore, we have*

$$\dim N(T-S) = \dim \left(N(T) \cap \bigcap_{n=1}^{\infty} T^n(X) \right).$$

(b) *If 0 is a boundary point of $\sigma(T)$, then 0 is a pole of the resolvent of T.*

For the proof of Theorem 1 we need some additional notation and a preliminary lemma.

Let $T \in \mathscr{L}(X)$. We write $\alpha(T)$ and $\beta(T)$ for $\dim N(T)$ and $\operatorname{codim} T(X)$, respectively. The operator T is called *lower semi-Fredholm* if $\beta(T)$ is finite (in this case T has closed range, by [4, Satz 55.4]). T is called *semi-Fredholm* if T is upper or lower semi-Fredholm. T is *Fredholm* if both $\alpha(T)$ and $\beta(T)$ are finite. The *index* of a semi-Fredholm operator T is defined by

$$\operatorname{ind}(T) = \alpha(T) - \beta(T).$$

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For $T \in \mathscr{L}(X)$ we define the number $k_n(T)$ by

$$k_n(T) = \dim((N(T) \cap T^n(X))/(N(T) \cap T^{n+1}(X))) \qquad (n \geq 0).$$

We say that T has *uniform descent* for $n \geq d$ if $k_n(T) = 0$ for $n \geq d$. This notion is due to Grabiner (see [3, Definition 1.3 and Lemma 2.3]). If T has uniform descent for $n \geq d$ and if $T^n(X)$ is closed in the operator range topology of $T^d(X)$ for $n \geq d$, then we say that T has *topological uniform descent* for $n \geq d$ (see [3, Definition 2.5]). For a discussion of operator ranges and their topologies, the reader is referred to [1] or [2].

Lemma. *Suppose that $T \in \mathscr{L}(X)$, T^2 has closed range, and $\dim T(X) \cap N(T) < \infty$. Then:*

(a) *There exists an integer $d \geq 0$ such that T has uniform descent for $n \geq d$.*
(b) *T has topological uniform descent for $n \geq d$ and $\bigcap_{n=1}^{\infty} T^n(X)$ is closed.*
(c) *$T(\bigcap_{n=1}^{\infty} T^n(X)) = \bigcap_{n=1}^{\infty} T^n(X)$.*
(d) *If $S \in \mathscr{L}(X)$ is invertible, $TS = ST$, and $\|S\|$ is sufficiently small, then $T - S$ has closed range.*

Proof. (a) Since

$$N(T) \cap T^{n+1}(X) \subseteq N(T) \cap T^n(X) \subseteq N(T) \cap T(X) \quad \text{for } n \geq 0$$

and $\dim(N(T) \cap T(X)) < \infty$, the result follows.

(b) Invoke [3, Lemma 2.4]. The hypotheses of this lemma are satisfied because of [3, Lemma 2.3].

(c) Use (b) and [3, Theorem 3.4(a)].

(d) Put $V = T - S$. Since T has topological uniform descent for $n \geq d$, V has closed range, by [3, Theorem 4.7(a)]. □

Proof of Theorem 1. (a) Put $X_0 = \bigcap_{n=1}^{\infty} T^n(X)$, and denote the restriction of T to X_0 by T_0. Clearly,

$$\alpha(T_0) = \dim\left(N(T) \cap \bigcap_{n=1}^{\infty} T^n(X)\right) \leq \dim(N(T) \cap T(X)) < \infty.$$

Part (c) of the above lemma shows that $T_0(X_0) = X_0$; thus, $\beta(T_0) = 0$. It follows that T_0 is Fredholm with

$$\operatorname{ind}(T_0) = \alpha(T_0) - \beta(T_0) = \alpha(T_0).$$

[4, Satz 82.4] shows that there exists $\varepsilon > 0$ such that

$$T_0 - R \text{ is Fredholm}$$

and

$$\operatorname{ind}(T_0 - R) = \operatorname{ind}(T_0) \quad \text{for } R \in \mathscr{L}(X_0) \text{ with } \|R\| < \varepsilon.$$

Furthermore, again by [4, Satz 82.4],

$$\alpha(T_0 - R) \leq \alpha(T_0), \qquad \beta(T_0 - R) \leq \beta(T_0) = 0$$

for $R \in \mathscr{L}(X_0)$ with $\|R\| < \varepsilon$. This gives $\beta(T_0 - R) = 0$ and

$$(*) \qquad \alpha(T_0 - R) = \operatorname{ind}(T_0 - R) = \operatorname{ind}(T_0) = \alpha(T_0)$$

for each $R \in \mathscr{L}(X_0)$ such that $\|R\| < \varepsilon$.

Now suppose that $S \in \mathscr{L}(X)$ is invertible, $TS = ST$, and $\|S\| < \varepsilon$. Since S commutes with T, we have $S(X_0) \subseteq X_0$. Put $S_0 = S|X_0$; then $S_0 \in \mathscr{L}(X_0)$ and $\|S_0\| < \varepsilon$.

Next we show that $N(T - S) = N(T_0 - S_0)$. The inclusion $N(T_0 - S_0) \subseteq N(T-S)$ is clear. Let $x \in N(T-S)$; thus, $Tx = Sx$. Since $TS = ST$, we have $T^n x = S^n x$ for each $n \in \mathbb{N}$; therefore, $x = S^{-n}T^n x = T^n(S^{-n}x) \in T^n(X)$ for each $n \in \mathbb{N}$ and, hence, $x \in X_0$. This proves $N(T - S) = N(T_0 - S_0)$. Therefore, by $(*)$,

$$\alpha(T - S) = \alpha(T_0 - S_0) = \alpha(T_0).$$

To complete the proof of (a), we have to show that $T - S$ has closed range for sufficiently small $\|S\|$. But this follows immediately from part (d) of the above lemma.

(b) follows from [3, Corollary 4.9]. □

Remark. Theorem 1 can be proved under the weaker assumption that $T^{-1}(T^2(X)) = T(X) + N(T)$ is closed. $T(X) + N(T)$ being closed is equivalent to $T^2(X)$ is closed in the operator range topology on $T(X)$ (see the proof of [3, Theorem 3.2]), so our results are still true except that $X_0 = \bigcap_{n=1}^{\infty} T^n(X)$ is only known to be closed in the operator range topology on $T(X)$.

For $T \in \mathscr{L}(X)$ we define two *essential spectra*:

$$\sigma_e(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \text{ is not Fredholm}\}$$

and

$$\sigma_w(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \text{ is not a Fredholm operator with } \operatorname{ind}(T - \lambda I) = 0\}.$$

Theorem 2. *Let $T \in \mathscr{L}(X)$, and suppose that $T^2(X)$ is closed and* $\dim N(T) \cap T(X) < \infty$. *Then:*

(a) *If 0 is a boundary point of $\sigma_e(T)$, 0 is isolated in $\sigma_e(T)$.*
(b) *If 0 is a boundary point of $\sigma_w(T)$, 0 is isolated in $\sigma_w(T)$.*

Proof. Since T has topological uniform descent for $n \geq d$ (Lemma, part (b)), we can define

$$\alpha^*(T) = \lim_{n \to \infty} \dim N(T^{n+1})/N(T^n)$$

and

$$\beta^*(T) = \lim_{n \to \infty} \dim T^n(X)/T^{n+1}(X).$$

Theorem 4.7 in [3] then says that if $\lambda \in \mathbb{C} \setminus \{0\}$ is sufficiently small, then $\alpha(T - \lambda I) = \alpha^*(T - \lambda I) = \alpha^*(T)$ and $\beta(T - \lambda I) = \beta^*(T - \lambda I) = \beta^*(T)$. Thus, if any $T - \lambda I$ is semi-Fredholm for small $\lambda \neq 0$, then all are semi-Fredholm with the same index $\alpha^*(T) - \beta^*(T)$. This gives the results. □

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