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RANDOM APPROXIMATIONS AND RANDOM FIXED POINT THEOREMS FOR CONTINUOUS 1-SET-CONTRACTIVE RANDOM MAPS

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ABSTRACT. Recently the author [Proc. Amer. Math. Soc. **103** (1988), 1129–1135] proved random versions of an interesting theorem of Ky Fan [Theorem 2, Math. Z. **112** (1969), 234–240] for continuous condensing random maps and nonexpansive random maps defined on a closed convex bounded subset in a separable Hilbert space. In this paper, we prove that it is still true for (more general) continuous 1-set-contractive random maps, which include condensing, nonexpansive, locally almost nonexpansive (LANE), semicontractive maps, etc. Then we use these theorems to obtain random fixed points theorems for the above-mentioned maps satisfying weakly inward conditions. In order to obtain these results, we first need to prove a random fixed point theorem for 1-set-contractive self-maps in a separable Banach space. This leads to the discovery of some new random fixed point theorems in a separable uniform convex Banach space.

1. INTRODUCTION AND PRELIMINARIES

Random fixed point theory has received much attention for the last 16 years, since the publication of the survey article by Bharucha-Reid [2] in 1976. Random fixed point theorems are stochastic versions of (classical or deterministic) fixed point theorems, and are required for the theory of random equations. In this paper, we will consider a stochastic version of a very interesting theorem of Fan [6, Theorem 2] which is stated as follows.

Let K *be a nonempty compact convex set in a normed linear space* X. *For any continuous map* f *from* K *into* X, *there exists a point* u *in* K *such that*

$$\|u - f(u)\| = d(f(u), K),$$

where $d(f(u), K)$ *is the distance between* $f(u)$ *and* K.

Various nonstochastic aspects of this theorem has been studied by Fan [7], Ha [8], Lin [12, 13, 15–17], and Reich [22]. Stochastic aspects of the above

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theorem have been considered recently by Lin [14], Papageorgiou [19], Sehgal and Singh [23], and Sehgal and Walters [24]. In [14], we considered random versions of the above theorem for a continuous condensing random map defined on a closed ball in a separable Banach space, and either a continuous condensing random map or a nonexpansive random map defined on a closed convex bounded sets in a separable Hilbert space. In this paper, we will unify the above results by considering more general 1-set-contractive random maps, which include condensing nonexpansive maps, and other interesting maps such as locally almost nonexpansive (LANE), semicontractive, and the sum of a nonexpansive and a complete continuous map in a separable Hilbert space. We will use these theorems to obtain some new random fixed point theorems for the above-mentioned maps satisfying weakly inward conditions. These form §§3 and 4. In order to obtain these results, we need a random fixed point theorem for continuous 1-set-contractive self-maps (a map sends its domain into itself). As results of this, we prove several new random fixed point theorems for self-maps in a separable uniform convex Banach space, which is §2.

Now, we introduce our notation and definitions.

Let (Ω, Σ) be a measurable space with Σ a sigma algebra of subsets of Ω. Let X be a Banach space; a map $F\colon \Omega \to 2^X$ is called measurable if for each open subset B of X, $F^{-1}(B) \in \Sigma$, where 2^X is the family of all subsets of X, and $F^{-1}(B) = \{\omega \in \Omega | F(\omega) \cap B \text{ is not empty}\}$. This type of measurability is sometimes called weakly measurable (cf. [9]). Since we only consider this type of measurability in this paper, we just call it measurable as in [10]. We note that when $F(\omega) \in K(X)$ for all $\omega \in \Omega$, where $K(X)$ is the family of all nonempty compact subsets of X, F is measurable if and only if $F^{-1}(C) \in \Sigma$ for every closed subset C of X (see [9]). A measurable map $\varphi\colon \Omega \to X$ is called a measurable selector of a measurable map $F\colon \Omega \to \mathrm{CD}(X)$, if $\varphi(\omega) \in F(\omega)$ for each $\omega \in \Omega$, where $\mathrm{CD}(X)$ is the family of all nonempty closed subsets of X. Let S be a nonempty subset of X; let a map $f\colon \Omega \times S \to X$ is called a random operator if for each fixed $x \in S$ the map $f(\cdot, x)\colon \Omega \to X$ is measurable. A measurable map $\varphi\colon \Omega \to S$ is a random fixed point of a random operator f (or $F\colon \Omega \times S \to \mathrm{CD}(X)$) if $f(\omega, \varphi(\omega)) = \varphi(\omega)$ (or $\varphi(\omega) \in F(\omega, \varphi(\omega))$) for each $\omega \in \Omega$. A random operator $f\colon \Omega \times S \to X$ is called continuous (condensing, k-set-contractive, LANE, nonexpansive, completely continuous, semicontractive, etc.) if for each $\omega \in \Omega$, $f(\omega, \cdot)$ is continuous (condensing, k-set-contractive, LANE, nonexpansive, completely continuous, semicontractive, etc.). The definitions of condensing, k-set-contractive, LANE, nonexpansive, semicontractive, etc., will be defined below.

Let B be a nonempty bounded subset of a metric space X. Let $\gamma(\cdot)$ be the Kuratowski measure of noncompactness, i.e., $\gamma(B)$ be the infinum of the numbers r such that B can be covered by a finite number of subsets of X of diameter less than or equal to r. Let S be a nonempty subset of X, and let f be a continuous map from S into X. If for every nonempty bounded subset B of S with $\gamma(B) > 0$ we have $\gamma(f(B)) < \gamma(B)$, then f is called condensing. If there exists k, $0 \le k \le 1$, such that for each nonempty bounded subset B of S we have $\gamma(f(B)) \le k\gamma(B)$, then f is called k-set-contractive. It is clear that a k-set-contractive map, with $0 \le k < 1$, is a condensing map; and a condensing map is a 1-set-contractive map.

Let S be a nonempty subset of a normed space X and $f\colon S \to X$. Then f

is:

(1) nonexpansive if $\|f(x) - f(y)\| \leq \|x - y\|$ for each $x, y \in S$;

(2) completely continuous if it maps weakly convergent sequences into strongly convergent sequences;

(3) semicontractive (Browder [3]), if there exists a map $V: S \times S \to X$ such that $f(x) = V(x, x)$ for x in S, and

(a) For each fixed x in S, $V(\cdot, x)$ is nonexpansive from S to X;

(b) For each fixed x in S, $V(x, \cdot)$ is completely continuous from S to X, uniformly for u in a bounded subset of S (i.e., if v_j converges weakly to v in S, $\sup_{u\in B} \|V(u, v_j) - V(u, v)\| \to 0$ as $j \to \infty$, where $B \subseteq S$ is bounded);

(4) LANE (locally almost nonexpansive) (Nussbaum [18]) if f is continuous and for each $x \in S$ and $\varepsilon > 0$ there exists a weak neighborhood N_X of x in S (depending also on ε) such that $u, v \in N_x$, $\|f(u) - f(v)\| \leq \|u - v\| + \varepsilon$;

(5) demiclosed if $\{x_n\}$ is a sequence in S such that $x_n \to x \in S$ weakly and $f(x_n) \to y$ in X, then $f(x) = y$;

(6) demicompact if for every bounded sequence $\{x_n\}$ in S such that the sequence $\{x_n - f(x_n)\}$ is a convergent sequence in X, then there exists a strongly convergent subsequence of $\{x_n\}$.

For $x \in X$, let

$$p_s(x) = \{y \in S \mid \|x - y\| = d(x, S)\}, \quad \text{where } d(x, S) = \inf_{y\in S} \|x - y\|.$$

The set-valued map p_s is called the metric projection on S. If p_s is a single-valued map, it is called a proximity map. We will use p instead of p_s later.

Remark 1.1. For a continuous map $f: S \to X$, where S is bounded, it is easy to see that if f is demicompact, then $I - f$ is demiclosed. It is also known that every nonexpansive map is a 1-set-contractive map.

2. Random fixed point theorems for self-maps

Random fixed point theorems for continuous condensing or nonexpansive self-maps were proved by Itoh [10] in 1979. But for more general 1-set-contractive maps, no random fixed point theorem is yet available. The hard part is to find appropriate conditions for this type of map to have a fixed point. These conditions must be automatically (although nontrivial) satisfied by such useful maps as condensing, nonexpansive, LANE, semicontractive maps. Natural choices for these conditions for deterministic fixed point theorems are: $(I-f)(S)$ is closed, and S is a closed convex bounded set or a closed bounded set with nonempty interior (see [21] or [18]). For random fixed point theorems, these conditions often lead to failure. Now, in this section, we are able to finish this goal.

Theorem 2.1. *Let S be a nonempty weakly compact convex subset of a separable Banach space X, and let $f: \Omega\times S \to S$ be a continuous 1-set-contractive random operator such that $I - f(\omega, \cdot)$ is demiclosed, for each $\omega \in \Omega$, where I is the identity map on X.*

Then f has a random fixed point.

Proof. Motivated by Itoh [10, Theorem 2.5], take an element v in S and a sequence $\{k_n\}$ of real numbers such that $0 < k_n < 1$ and $k_n \to 0$, as $n \to \infty$.

For each n, define a mapping $f_n: \Omega \times S \to S$ by

$$f_n(\omega, x) = k_n v + (1 - k_n) f(\omega, x);$$

then f_n is a $(1 - k_n)$-set-contractive random operator. From [10, Theorem 2.1], f_n has a random fixed point φ_n, i.e., there exists a measurable map $\varphi_n: \Omega \to S$ such that $f_n(\omega, \varphi_n(\omega)) = \varphi_n(\omega)$ for all $\omega \in \Omega$. For each n, define $F_n: \Omega \to \mathrm{WK}(S)$ by

$$F_n(\omega) = \text{w-cl}\{\varphi_i(\omega) | i \geq n\},$$

where w-cl(C) denotes the weak closure of C and WK(S) denotes the family of all nonempty weakly compact subsets of S. Define $F: \Omega \to \mathrm{WK}(S)$ by

$$F(\omega) = \bigcap_{n=1}^{\infty} F_n(\omega).$$

Since S is weakly compact and X is separable, the weak topology on S is a metric topology (see [5, pp. 434]), S and $F_n(\omega)$ are compact in the weak topology from [9, Theorems 4.1 and 3.1], and F is w-measurable (i.e., F is measurable with respect to the weak topology on S). From the Kuratowski and Ryll-Nardzewski theorem [11], there is a w-measurable selector φ of F. For each $x^* \in X^*$ (the dual space of X), $x^*(\varphi(\cdot))$ is measurable as a numerically valued function in Ω. Since X is separable, φ is measurable (see, e.g. [1, pp. 14–16]). We will show that φ is a random fixed point of f. Fixing any ω in Ω, since $\varphi(\omega) \in F(\omega)$, there exists a subsequence $\{\varphi_m(\omega)\}$ of $\{\varphi_n(\omega)\}$ that converges weakly to $\varphi(\omega)$. From

$$\begin{aligned} \varphi_m(\omega) - f(\omega, \varphi_m(\omega)) &= k_m v + (1 - k_m) f(\omega, \varphi_m(\omega)) - f(\omega, \varphi_m(\omega)) \\ &= k_m(v - f(\omega, \varphi_m(w))), \end{aligned}$$

S is bounded, and $k_m \to 0$ as $m \to \infty$, $\{\varphi_m(\omega) - f(\omega, \varphi_m(\omega))\}$ converges to 0. By the demiclosedness of $I - f(\omega, \cdot)$, we have $f(\omega, \varphi(\omega)) = \varphi(\omega)$, i.e., φ is a random fixed point of f. □

Remark 2.1. As pointed out by the referee, the w-measurable of F, in the proof of Theorem 2.1, can also be justified as follows.

Note that by definition,

$$F(\omega) = \mathrm{w} - \overline{\lim} F_n(\omega) = \{z \in S | \underline{\lim}\, d(z, F_n(\omega)) = 0\},$$

with $d(\cdot, \cdot)$ being a metric whose topology coincides with the weak topology on S (such a metric exists since X is separable and S is weakly compact).

However, we feel that our arguments work equally well. Although we quote two theorems from [9], those two theorems are elementary properties of measurability of a function. One states when the countable intersections of functions are measurable, and the other when the different definitions of measurability of a function are equivalent. Actually, Himmelberg's paper [9] has become an essential reference for people working in this area because it gives all the necessary measurable relations for random fixed point theory.

Theorem 2.2. *Let S be a nonempty closed convex bounded subset of a separable uniform convex Banach space X, and let $f: \Omega \times S \to S$ be a LANE random operator.*

Then f has a random fixed point.

Proof. From [18, Lemma 1 and the proof of Lemma 3, pp. 762–763], f is a 1-set-contractive map and $I - f(\omega, \cdot)$ is demiclosed for each $\omega \in \Omega$. Since S is weakly compact, from Theorem 2.1, f has a random fixed point. □

Theorem 2.3. *Let S be a nonempty closed convex bounded subset of a separable uniform convex Banach space X, $g: \Omega \times S \to S$ be a LANE random operator, $h: \Omega \times S \to S$ be a completely econtinuous random map, and $f = g + h$.*

Then f has a random fixed point.

Proof. From [21, Remark 3.7] f is also a LANE map. From Theorem 2.2, f has a random fixed point. □

Corollary 2.1. *Let S be a nonempty closed convex bounded subset of a separable uniform convex Banach space X, and let $f: \Omega \times S \to S$ be a continuous semicontractive random operator.*

Then f has a random fixed point.

Proof. From [18, p. 761], f is also a LANE map. The corollary follows from Theorem 2.2. □

Corollary 2.2 (Itoh [10, Theorem 2.5]). *Let S be a nonempty closed convex bounded subset of a separable uniform convex Banach space X, $g: \Omega \times S \to S$ a nonexpansive random operator, $h: \Omega \times S \to S$ a completely continuous random map, and $f = g + h$.*

Then f has a random fixed point.

Proof. It is easy to see that f is a semicontractive map under the representation $V(u, v) = g(u) + h(v)$. The results follows from Corollary 2.1. □

Remark 2.2. Theorem 2.2 is a stochastic version of Nussbaum [18, Theorem 1, p. 764].

3. Random approximations

In this section, we will give random versions of all theorems in Lin and Yen's [17], §2 by using theorems in §2.

Theorem 3.1. *Let S be a nonempty closed convex bounded subset of a separable Hilbert space X, and let $f: \Omega \times S \to X$ be a continuous 1-set-contractive random operator such that $I - p \circ f(\omega, \cdot)$ is demiclosed, for each $\omega \in \Omega$, where I is the identity map on X and p is the proximity map from X into S.*

Then there exists a measurable map $\varphi: \Omega \to S$ such that

$$\|\varphi(\omega) - f(\omega, \varphi(\omega)\| = d(f(\omega, \varphi(\omega)), S), \quad \text{for all } \omega \in \Omega.$$

Proof. Since p is nonexpansive in a Hilbert space [4], it is easy to see that $p \circ f$ is a continuous 1-set-contractive random map from $\Omega \times S$ into S. Since S is weakly compact, from Theorem 2.1, there is a measurable map $\varphi: \Omega \to S$ such that $p \circ f(\omega, \varphi(\omega)) = \varphi(\omega)$ for all $\omega \in \Omega$. Hence,

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = \|p \circ f(\omega, \varphi(\omega)) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega), S)$$

for all $\omega \in \Omega$. □

Corollary 3.1 (Lin [14, Theorem 2]). *Let S be a nonempty closed convex bounded subset of a separable Hilbert space X, and let $f: \Omega \times S \to X$ be a continuous condensing random operator.*

Then there exists a measurable map $\varphi: \Omega \to S$ *such that*

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega)), S) \quad \textit{for all } \omega \in \Omega.$$

Proof. Let p be the proximity map from X into S. Since p is nonexpansive [4], $p \circ f$ is also a continuous condensing and, therefore, 1-set-contractive random operator. For each $\omega \in \Omega$, from [20, p. 321], $p \circ f(\omega, \cdot)$ is demicompact; therefore, $I - p \circ f(\omega, \cdot)$ is demiclosed (cf. Remark 1.1). From Theorem 3.1, we have the corollary. □

Corollary 3.2 (Lin [14, Theorem 3]). *Let* S *be a nonempty closed convex bounded subset of a separable Hilbert space* X, *and let* $f: \Omega \times S \to X$ *be a nonexpansive random operator.*

Then there exists a measurable map $\varphi: \Omega \to S$ *such that*

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega)), S) \quad \textit{for all } \omega \in \Omega.$$

Proof. Let p be the proximity map from X into S. Since p is nonexpansive, $p \circ f$ is also nonexpansive and, from [3], $I - p \circ f(\omega, \cdot)$ is demiclosed, for each $\omega \in \Omega$. From Theorem 3.1, we have the corollary. □

Theorem 3.2. *Let* S *be a nonempty closed convex bounded subset of a separable Hilbert space* X, *and let* $f: \Omega \times S \to X$ *be a LANE random operator.*

Then there exists a measurable map $\varphi: \Omega \to S$ *such that*

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega)), S) \quad \textit{for all } \omega \in \Omega.$$

Proof. Let p be the proximity map from X into S. From the nonexpansiveness of p, it is easy to see that $p \circ f$ is also a LANE map, and therefore a 1-set-contractive map; from [18, pp. 762–763], $I - p \circ f(\omega, \cdot)$ is demiclosed. From Theorem 3.1, we have the theorem. □

Theorem 3.3. *Let* S *be a nonempty closed convex bounded subset of a separable Hilbert space* X, $g: \Omega \times S \to X$ *a LANE random operator,* $h: \Omega \times S \to X$ *a completely continuous random map, and* $f = g + h$.

Then there exists a measurable map $\varphi: \Omega \to S$ *such that*

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega)), S) \quad \textit{for all } \omega \in \Omega.$$

Proof. From [21, Remark 3.7], f is also a LANE map. From Theorem 3.2, we have the theorem. □

Corollary 3.3. *Let* S *be a nonempty closed convex bounded subset of a separable Hilbert space* X, *and let* $f: \Omega \times S \to X$ *be a continuous semicontractive random operator.*

Then there exists a measurable map $\varphi: \Omega \to S$ *such that*

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega)), S) \quad \textit{for all } \omega \in \Omega.$$

Proof. From [18, p. 761], f is also a LANE map. The results follows from Theorem 3.2. □

Corollary 3.4. *Let* S *be a nonempty closed convex bounded subset of a separable Hilbert space* X, $g: \Omega \times S \to X$ *a nonexpansive random operator,* $h: \Omega \times S \to X$ *a completely random map, and* $f = g + h$.

Then there exists a measurable map $\varphi: \Omega \to S$ *such that*

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega)), S) \quad \textit{for all } \omega \in \Omega.$$

Proof. It is easy to see that f is a semicontractive map under the representation $V(u, v) = g(u) + h(v)$. The results follows from Corollary 3.3. □

Remark 3.1. Corollary 3.2 can be viewed as a special case of Corollary 3.4 by letting h be identically equal to zero. Actually, Lin [14] proved Corollary 3.1 for the case requiring only that $f(\omega, S)$ be bounded for each $\omega \in \Omega$ instead of S being bounded. But for most of the work on deterministic fixed point theorems for LANE, nonexpansive, semicontractive maps, S is assumed bounded (see, e.g., [21, 18]).

4. Random fixed point theorems for non-self-maps

In this section, we will prove some new random fixed point theorems for non-self-maps, by using Theorem 3.1. These theorems are random versions of theorems in Lin and Yen's [17, §3].

Theorem 4.1. *Let S be a nonempty closed convex bounded subset of a separable Hilbert space X, and let $f: \Omega\times S \to X$ be a continuous 1-set-contractive random operator such that $I - p \circ f(\omega, \cdot)$ is demiclosed, for each $\omega \in \Omega$, where I is the identity map on X and p is the proximity map from X into S. Moreover, f satisfies any one of the following conditions:*

(i) *For each $\omega \in \Omega$ and each $x \in S$, there exists a number λ (real or complex, depending on whether the vector space X is real or complex) such that $|\lambda| < 1$ and $\lambda x + (1 - \lambda)f(\omega, x) \in S$.*

(ii) *For each $\omega \in \Omega$ and each $x \in S$ with $x \neq f(\omega, x)$, there exists y, depending on ω and x, in $I_S(x) = \{x + c(z - x) |\ \text{some}\ z \in S,\ c > 0\}$ such that $\|y - f(\omega, x)\| < \|x - f(\omega, x)\|$.*

(iii) *f is weakly inward (i.e., for each $\omega \in \Omega$, $f(\omega, x) \in \operatorname{cl} I_S(x)$ for $x \in S$).*
Then f has a random fixed point.

Proof. From Theorem 3.1, there exists a measurable map $\varphi: \Omega \to S$ such that

$$\|\varphi(\omega) - f(\omega, \varphi(\omega))\| = d(f(\omega, \varphi(\omega)), S) \quad \text{for all } \omega \in \Omega.$$

We will prove that φ is the desired random fixed point if f satisfies any one of the above conditions.

Let f satisfy condition (i). If φ is not a random fixed point of f, then there exists $\omega \in \Omega$ such that $\varphi(\omega) \neq f(\omega, \varphi(\omega))$. To this $\varphi(\omega) \in S$, there exists a number λ such that $|\lambda| < 1$ and $\lambda\varphi(\omega) + (1 - \lambda)f(\omega, \varphi(\omega)) = x \in S$. Therefore,

$$\begin{aligned} 0 < \|\varphi(\omega) - f(\omega, \varphi(\omega))\| &= d(f(\omega, \varphi(\omega)), S) \leq \|x - f(\omega, \varphi(\omega))\| \\ &= |\lambda| \|\varphi(\omega) - f(\omega, \varphi(\omega))\| < \|\varphi(\omega) - f(\omega, \varphi(\omega))\|, \end{aligned}$$

which is a contradiction. Hence φ is a random fixed point of f.

Let f satisfy condition (ii). If φ is not a random fixed point of f, then there exists $\omega \in \Omega$ such that $\varphi(\omega) \neq f(\omega, \varphi(\omega))$. From the assumption (ii), there exists y in $I_S(\varphi(\omega))$ such that

$$\|y - f(\omega, \varphi(\omega))\| < \|\varphi(\omega) - f(\omega, \varphi(\omega))\|.$$

Since $y \in I_S(\varphi(\omega))$, there exists $z \in S$, $c > 0$ such that $y = \varphi(\omega) + c(z - \varphi(\omega))$. Since y is not in S and otherwise contradicts the choice of φ, we can assume that $c > 1$. Then $z = y/c + (1 - 1/c)\varphi(\omega) = (1 - \beta)y + \beta\varphi(\omega)$, where $\beta = 1 - 1/c$, $0 < \beta < 1$. Therefore,

$$\begin{aligned}\|z - f(\omega, \varphi(\omega))\| &\leq (1 - \beta)\|y - f(\omega, \varphi(\omega))\| + \beta\|\varphi(\omega) - f(\omega, \varphi(\omega))\| \\ &< (1 - \beta)\|\varphi(\omega) - f(\omega, \varphi(\omega))\| + \beta\|\varphi(\omega) - f(\omega, \varphi(\omega))\| \\ &= \|\varphi(\omega) - f(\omega, \varphi(\omega))\|\end{aligned}$$

which contradicts the choice of φ. Hence, $f(\omega, \varphi(\omega)) = \varphi(\omega)$ for all $\omega \in \Omega$ and φ is the desired random fixed point of f.

Let f satisfy condition (iii). For each $\omega \in \Omega$ and each $x \in S$ with $x \neq f(\omega, x)$, since $f(\omega, x) \in \operatorname{cl} I_S(x)$, there exists y in $I_S(x)$ such that $\|y - f(\omega, x)\| < \|x - f(\omega, x)\|$ and f satisfies condition (ii). □

Corollary 4.1 (Lin [14, Theorem 5]). *Let S be a nonempty closed convex bounded subset of a separable Hilbert space X, and let $f: \Omega \times S \to X$ be a continuous condensing random operator. Moreover, f satisfies any one of the conditions* (i)–(iii) *in Theorem* 4.1.

Then f has a random fixed point.

Proof. Let p be the proximity map from X into S. Since p is nonexpansive, $p \circ f$ is also a continuous condensing and therefore 1-set-contractive random operator. For each $\omega \in \Omega$, from [20, p. 321], $p \circ f(\omega, \cdot)$ is demicompact, therefore $I - p \circ f(\omega, \cdot)$ is demiclosed (cf. Remark 1.1). From Theorem 4.1, we have the corollary. □

We note that the proof of Corollary 4.1 is almost identical to the proof of Corollary 3.1, replacing Theorem 3.1 by Theorem 4.1. Similarly, we have the following Corollaries 4.2–4.4 and Theorems 4.2–4.3. The proofs of them are almost identical to Corollaries 3.2–3.4 and Theorems 3.2–3.3. Therefore, we will state them without proof.

Corollary 4.2 (Lin [14, Theorem 6]). *Let S be a nonempty closed convex bounded subset of a separable Hilbert space X, and let $f: \Omega \times S \to X$ be a nonexpansive random operator. Moreover, f satisfies any one of the conditions* (i)–(iii) *in Theorem* 4.1.

Then f has a random fixed point.

Theorem 4.2. *Let S be a nonempty closed convex bounded subset of a separable Hilbert space X, and let $f: \Omega \times S \to X$ be a LANE random operator. Moreover, f satisfies any one of the conditions* (i)–(iii) *in Theorem* 4.1.

Then f has a random fixed point.

Theorem 4.3. *Let S be a nonempty closed convex bounded subset of a separable Hilbert space X, $g: \Omega \times S \to X$ a LANE random operator, $h: \Omega \times S \to X$ a completely continuous random map, and $f = g + h$. Moreover, f satisfies any one of the conditions* (i)–(iii) *in Theorem* 4.1.

Then f has a random fixed point.

Corollary 4.3. *Let S be a nonempty closed convex bounded subset of a separable Hilbert space X, and let $f: \Omega \times S \to X$ be a continuous semicontractive random operator. Moreover, f satisfies any one of the conditions* (i)–(iii) *in Theorem* 4.1.

Then f has a random fixed point.

Corollary 4.4. *Let S be a nonempty closed convex bounded subset of a separable Hilbert space X, $g: \Omega \times S \to X$ a nonexpansive random operator, $h: \Omega \times S \to X$ a completely continuous random operator, and $f = g + h$. Moreover, f satisfies any one of the conditions* (i)–(iii) *in Theorem* 4.1.

Then f has a random fixed point.

Remark 4.1. Xu [25] extended Corollary 4.1 to a separable Banach space. If S has a nonempty interior, he also extended Corollary 4.2 to a uniformly convex Banach space. The other theorems and corollaries in this section are all new. Actually, this paper seems to be the first to prove some random fixed point theorems for continuous 1-set-contractive maps with success. This means that we are able to prove random fixed point theorems for those interesting maps—LANE, semicontractive—without putting additional conditions on those maps.

Remark 4.2. All the theorems and corollaries in §4 remain true if f satisfies any one of the following conditions:

(iv) For each $\omega \in \Omega$ and any u on the boundary of S with $u = p(f(\omega, u))$, u is a fixed point of $f(\omega, \cdot)$.

(v) For each $\omega \in \Omega$ and each x on the boundary of S, $\|f(\omega, x) - y\| \leq \|x - y\|$ for some $y \in S$.

The proof of this is a modification of Theorem 4.1 and Theorem 5 of [17]. We will omit this proof, because these conditions do not seem to be as popular as conditions (i)–(iii) in Theorem 4.1. With this remark, we finish the random versions of all the theorems of Lin and Yen's [17, §3].

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