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OPERATORS WITH COMPLEX GAUSSIAN KERNELS: BOUNDEDNESS PROPERTIES

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ABSTRACT. Boundedness properties are stated for some operators from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$, $1 \le p, q \le \infty$, with complex Gaussian kernels. Their contraction properties are also analysed.

1. INTRODUCTION

In this paper we will study boundedness properties for the general complex Gaussian operator in dimension one,

$$(1.1)\quad (\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(x) = \int_{-\infty}^{+\infty} \exp\{-\beta x^2 - \varepsilon y^2 + 2\delta xy + \xi x + \gamma y\} \cdot f(y)\, dy,$$

$\beta, \varepsilon, \delta, \xi, \gamma \in \mathbb{C}$, $x \in \mathbb{R}$, from the space of complex-valued functions $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$, $1 \le p, q \le \infty$, relative to the Lebesgue measure.

This subject was originally of interest in the context of Quantum Field Theory (see [1]). The complex Gaussian operator (1.1) has an intrinsic interest due to the basic role of the extended oscillator semigroup introduced by Howe [4] (see also Folland [3, Chapter 5]).

In his important paper [5], Lieb extends the operator (1.1) to n dimensions and develops an extensive study of (1.1) in the context of the spaces $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Moreover, he generalizes the results given by Epperson in §2 of [2] for (1.1). Lieb stated that for the nondegenerate case, that is, $(\operatorname{Re}\delta)^2 < (\operatorname{Re}\beta)\cdot(\operatorname{Re}\varepsilon)$, $\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ has exactly one maximizer which is a centered Gaussian function e^{sy^2}, $s \in \mathbb{C}$. For the degenerate case, that is, $(\operatorname{Re}\delta)^2 = (\operatorname{Re}\beta)\cdot(\operatorname{Re}\varepsilon)$, the question of the existence of a maximizer is a subtle one. This problem requires a complicated algebraic study, and precise conditions are not given there.

The purpose of our paper is to calculate the *exact* region of boundedness of (1.1) for each $1 \le p, q \le \infty$, in both degenerate and nondegenerate cases.

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The proof of this result follows a technique initiated by Weissler in Theorem 1 of [6]. Weissler sated the exact region of boundedness for the Hermite semigroup.

We conclude this paper by giving sufficient conditions in both degenerate and nondegenerate cases for the operator $\mathcal{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$, $\beta, \varepsilon, \delta, \xi, \gamma \in \mathbb{R}$, to be a contraction over $L^p(\mathbb{R})$, $1 < p < \infty$, and a contraction as an operator from $L^2(\mathbb{R})$ into $L^p(\mathbb{R})$, $0 < p < \infty$. These results follow those given in Chapter 8 of [1].

Throughout this paper we will take square roots with positive real part.

2. Boundedness properties

For $1 \leq p \leq \infty$, let $(I_p f)(x) = (2\pi)^{-1/2p} \cdot \exp(-x^2/2p) \cdot f(x)$. For $z \in \mathbb{C}$, let $(Q_z f)(x) = e^{zx} \cdot f(x)$. For $\gamma^* > 0$, let $(T_{\gamma^*} f)(x) = f(\gamma^* x)$. Also, for $\alpha^* \in \mathbb{C}$, let $(M_{\alpha^*} f)(x) = \exp(-\alpha^* x^2/2) \cdot f(x)$. The Gauss-Weierstrass semigroup on $\mathbb{R}$ is given by

$$(e^{z\Delta} f)(x) = (4\pi z)^{-1/2} \cdot \int_{-\infty}^{+\infty} \exp[-(x-y)^2/4z] \cdot f(y)\, dy, \qquad x \in \mathbb{R},$$

where $\operatorname{Re} z \geq 0$ (and $z \neq 0$). Finally, we denote by $\|\mathcal{F}_{\beta,\varepsilon,\delta,\varepsilon,\gamma}\|_{p,q}$ the norm of $\mathcal{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ as a map from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$, $1 \leq p, q \leq \infty$.

It is easy to check that for any $\gamma^* > 0$ and $\operatorname{Re}\delta \geq 0$,

$$\mathcal{F}_{\beta,\varepsilon,\delta,\xi,\gamma} = (\pi\gamma^*/\delta)^{1/2} (2\pi)^{(1/2p)-(1/2q)} Q_\xi I_q^{-1} M_{\beta^*} T_{\gamma^*} e^{(\gamma^*/4\delta)\Delta} M_{\alpha^*} I_p Q_\gamma, \tag{2.1}$$

where

$$\alpha^* = 2\varepsilon - \frac{1}{p} - \frac{2\delta}{\gamma^*}, \qquad \beta^* = 2\beta + \frac{1}{q} - 2\gamma^*\delta.$$

We will denote by q' the exponent conjugate to q.

Theorem 2.1. *The following hold:*

(i) *If* $1 \leq p \leq q \leq \infty$, $\operatorname{Re}\delta > 0$, *and* $\operatorname{Re}\varepsilon > 0$, *then* $\mathcal{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ *is bounded from* $L^p(\mathbb{R})$ *to* $L^q(\mathbb{R})$ *if and only if*

$$(\operatorname{Re}\varepsilon) \cdot (\operatorname{Re}\beta) \geq (\operatorname{Re}\delta)^2.$$

(ii) *If* $1 \leq q < p \leq \infty$, $\operatorname{Re}\delta > 0$, *and* $\operatorname{Re}\varepsilon > 0$, *then* $\mathcal{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ *is bounded from* $L^p(\mathbb{R})$ *to* $L^q(\mathbb{R})$ *if and only if*

$$(\operatorname{Re}\varepsilon) \cdot (\operatorname{Re}\beta) > (\operatorname{Re}\delta)^2.$$

(iii) *If* $1 \leq p, q \leq \infty$, $\operatorname{Re}\delta = 0$, $\frac{1}{4} - \frac{1}{2p} \leq \operatorname{Re}\varepsilon \leq \frac{1}{2} - \frac{1}{2p}$, *and* $\operatorname{Re}\beta \geq \operatorname{Re}\varepsilon + \frac{1}{2p} - \frac{1}{2q'} \geq 0$, *then* $\mathcal{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ *is bounded from* $L^p(\mathbb{R})$ *to* $L^q(\mathbb{R})$ *if and only if* $\operatorname{Re}\varepsilon \geq 0$.

(iv) *If* $1 \leq p, q \leq \infty$, $\operatorname{Re}\delta = 0$, $\frac{1}{2p} - \frac{1}{2q'} \leq \operatorname{Re}\beta \leq \operatorname{Re}\varepsilon + \frac{1}{2p} - \frac{1}{2q'}$, *and* $\frac{1}{4} - \frac{1}{2q'} \leq \operatorname{Re}\beta \leq \frac{1}{2} - \frac{1}{2q'}$, *then* $\mathcal{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ *is bounded from* $L^p(\mathbb{R})$ *to* $L^q(\mathbb{R})$ *if and only if* $\operatorname{Re}\varepsilon \geq 0$ *and* $\operatorname{Re}\beta \geq 0$.

Proof. Suppose first that $\operatorname{Re}\delta = 0$, $\frac{1}{4} - \frac{1}{2p} \leq \operatorname{Re}\varepsilon \leq \frac{1}{2} - \frac{1}{2p}$, $\operatorname{Re}\beta \geq \operatorname{Re}\varepsilon + \frac{1}{2p} - \frac{1}{2q'} \geq 0$, and $1 \leq p, q \leq \infty$. Now, if $\operatorname{Re}\varepsilon \geq 0$, $\operatorname{Re}\beta \geq 0$, and m is a real number such that $\operatorname{Re} 2\varepsilon + \frac{1}{p} = \frac{1}{m}$, we have $1 \leq m \leq 2$, and the Hausdorff-Young inequality yields that $e^{(\gamma^*/4\delta)\Delta}$ is bounded from $L^m(\mathbb{R})$ into $L^{m'}(\mathbb{R})$ (m' is the exponent conjugate to m). But from the hypothesis $\mathscr{F}_{\beta-(1/2q),\varepsilon+(1/2p),\delta,\xi,\gamma}$ is bounded from $L^m(\mathbb{R}, e^{-x^2/2}\,dx)$ into $L^{m'}(\mathbb{R}, e^{-x^2/2}dx)$. Since $p \geq m$ and $q \leq m'$, the operator $\mathscr{F}_{\beta-(1/2q),\varepsilon+(1/2p),\delta,\xi,\gamma}$ is bounded from $L^p(\mathbb{R}, e^{-x^2/2}dx)$ into $L^q(\mathbb{R}, e^{-x^2/2}dx)$, $1 \leq p, q \leq \infty$, and therefore $\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$.

For the case $\operatorname{Re}\delta = 0$, $\frac{1}{2p} - \frac{1}{2q'} \leq \operatorname{Re}\beta \leq \operatorname{Re}\varepsilon + \frac{1}{2p} - \frac{1}{2q'}$, $\frac{1}{4} - \frac{1}{2q'} \leq \operatorname{Re}\beta \leq \frac{1}{2} - \frac{1}{2q'}$, and $1 \leq p, q \leq \infty$, the proof is similar.

Now assume $\operatorname{Re}\delta > 0$, so that $\operatorname{Re}(\gamma^*/4\delta) > 0$ and therefore, for $1 \leq p \leq q \leq \infty$, $e^{(\gamma^*/4\delta)\Delta}$ is bounded from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$. Since $\operatorname{Re}\varepsilon \geq 0$, $\operatorname{Re}\beta \geq 0$, and $(\operatorname{Re}\varepsilon)\cdot(\operatorname{Re}\beta) \geq (\operatorname{Re}\delta)^2$, γ^* can be chosen so that $\operatorname{Re}\alpha^* \geq 0$ and $\operatorname{Re}\beta^* \geq 0$. It follows from (2.1) that $\mathscr{F}_{\beta-(1/2q),\varepsilon+(1/2p),\delta,\xi,\gamma}$ is bounded from $L^p(\mathbb{R}, e^{-x^2/2}\,dx)$ into $L^q(\mathbb{R}, e^{-x^2/2}\,dx)$, $1 \leq p, q \leq \infty$, and hence $\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$.

For the case $1 \leq q < p \leq \infty$ and from the conditions $\operatorname{Re}\varepsilon \geq 0$, $\operatorname{Re}\beta \geq 0$, and $(\operatorname{Re}\varepsilon)\cdot(\operatorname{Re}\beta) > (\operatorname{Re}\delta)^2$, γ^* can be chosen so that $\operatorname{Re}\alpha^* > 0$ and $\operatorname{Re}\beta^* > 0$. Observing that M_{β^*} is a bounded map from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$ for $\operatorname{Re}\beta^* > 0$ and since $e^{(\gamma^*/4\delta)\Delta}$ is bounded over $L^p(\mathbb{R})$, equality (2.1) implies that $\mathscr{F}_{\beta-(1/2q),\varepsilon+(1/2p),\delta,\xi,\gamma}$ is bounded from $L^p(\mathbb{R}, e^{-x^2/2}dx)$ into $L^q(\mathbb{R}, e^{-x^2/2}\,dx)$, $1 \leq q < p \leq \infty$. Therefore, $\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$.

In order to prove the converse, suppose $\|\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}\|_{p,q} < \infty$. We will prove that $\operatorname{Re}\varepsilon \geq 0$, $\operatorname{Re}\beta \geq 0$, and $(\operatorname{Re}\varepsilon)\cdot(\operatorname{Re}\beta) \geq (\operatorname{Re}\delta)^2$ and that $(\operatorname{Re}\varepsilon)\cdot(\operatorname{Re}\beta) > (\operatorname{Re}\delta)^2$ holds if $q < p$. To this end, we need to calculate the action of $\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ on an arbitrary Gaussian function $g_s(y) = e^{sy^2}$, $s \in \mathbb{C}$, $y \in \mathbb{R}$. Then $\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ can be computed for $\operatorname{Re}s < \operatorname{Re}\varepsilon$ to obtain

(2.2)
$$(\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}g_s)(x) = \left(\frac{\pi}{\varepsilon - s}\right)^{1/2} \cdot \exp\left(\frac{\delta^2 - \beta\varepsilon + \beta s}{\varepsilon - s}x^2 + \frac{\delta\gamma + \xi\varepsilon - \xi s}{\varepsilon - s}x + \frac{\gamma^2}{4(\varepsilon - s)}\right),$$

with $x \in \mathbb{R}$.

We impose that $g_s \in L^p(\mathbb{R})$ so that $\operatorname{Re}s < 0$. Now, we want (2.2) to be in $L^q(\mathbb{R})$. With this purpose, let us consider the transformation $L(s)$, given by

$$L(s) = \frac{\delta^2 - \beta\varepsilon + \beta s}{\varepsilon - s}.$$

If $\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma} \in L^q(\mathbb{R})$, then $\operatorname{Re}L(s) < 0$. Note that L maps ε to ∞. Therefore, given the line $\operatorname{Re}s = 0$, there exists a circle C passing through ε such that L applies C into the line $\operatorname{Re}s = 0$. We claim that $\operatorname{Re}\varepsilon \geq 0$. In fact, assume that $\operatorname{Re}\varepsilon < 0$. Then $\operatorname{Re}s < \operatorname{Re}\varepsilon < 0$. Let s_0 be a point of the circle C satisfying $\operatorname{Re}s_0 < 0$ and $\operatorname{Re}L(s_0) = 0$. Assume that $s \to s_0$ with the restrictions $\operatorname{Re}s \leq -\varepsilon^*$ $(\varepsilon^* > 0)$ and $\operatorname{Re}L(s) < 0$. Then g_s remains bounded in $L^p(\mathbb{R})$ as $s \to s_0$, while $\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ blows up in $L^q(\mathbb{R})$. This is a contradiction because $\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ is bounded, and we conclude that $\operatorname{Re}\varepsilon \geq 0$.

In order to verify that $\operatorname{Re}\beta \geq 0$, let

$$L_1(s) = L(s) + \beta = \frac{\delta^2}{\varepsilon - s}. \tag{2.3}$$

Taking $\operatorname{Re} s = -\varepsilon^*$ $(\varepsilon^* > 0)$ and noting that $\operatorname{Re} s < 0$ implies $\operatorname{Re} L(s) < 0$, we have

$$\operatorname{Re} L_1(s) = \operatorname{Re} L(s) + \operatorname{Re}\beta \le \operatorname{Re}\beta. \tag{2.4}$$

Now, letting s tend to infinity along the line $\operatorname{Re} s = -\varepsilon^*$, we see that $0 \le \lim_{s\to s_0} \operatorname{Re} L_1(s) \le \operatorname{Re}\beta$, whence $\operatorname{Re}\beta \ge 0$. Thus $\operatorname{Re}\varepsilon \ge 0$ and $\operatorname{Re}\beta \ge 0$.

Next suppose that $\operatorname{Re}\varepsilon > 0$. In this case L_1 carries the line $\operatorname{Re} s = 0$ into a circle C_1 passing through 0. By (2.4), to prove that $(\operatorname{Re}\varepsilon)\cdot(\operatorname{Re}\beta) \ge (\operatorname{Re}\delta)^2$, it suffices to show that some point on that circle has real part

$$[\operatorname{Re}\delta]^2 \cdot [\operatorname{Re}\varepsilon]^{-1}.$$

Denote such a point by s_1. From (2.4) we obtain

$$\operatorname{Re} L_1(s_1) \le \operatorname{Re}\beta,$$

so that $(\operatorname{Re}\varepsilon)\cdot(\operatorname{Re}\beta) \ge (\operatorname{Re}\delta)^2$. The center of the circle C_1 is $\frac{1}{2}L_1(s_2)$, where s_2 minimizes $|\varepsilon - s|$ subject to the condition $\operatorname{Re} s = 0$. It is easy to check that $\frac{1}{2}L_1(s_2) + |\frac{1}{2}L_1(s_2)|$ has the desired real part.

Finally, if $q < p$, we must show that the equality cannot hold in $(\operatorname{Re}\varepsilon)\cdot(\operatorname{Re}\beta) > (\operatorname{Re}\delta)^2$. Indeed, if it did, then γ^* could be chosen so that $\operatorname{Re}\alpha^* = \operatorname{Re}\beta^* = 0$ in (2.1). Then (2.1) would imply that $e^{(\gamma^*/4\delta)\Delta}$, with $\operatorname{Re}(\gamma^*/4\delta) > 0$, is bounded from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$, which is false. □

3. Contraction properties

The purpose of this section is to give sufficient conditions in order that the operator $\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}$ (for degenerate and nondegenerate cases), $\beta, \varepsilon, \delta, \xi, \gamma \in \mathbb{R}$, be a contraction over $L^p(\mathbb{R})$, $1 < p < \infty$, and a contraction as an operator from $L^2(\mathbb{R})$ into $L^p(\mathbb{R})$, $0 < p < \infty$.

The next results are motivated by Chapter 8 of [1].

Theorem 3.1. *Let $1 < p < \infty$, and assume $\varepsilon > 0$ and $\delta^2 < p'\beta\varepsilon$ (here, p' denotes the exponent conjugate to p). For all $f \in L^p(\mathbb{R})$, we have*

(3.1)
$$\begin{aligned}\|\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_p^p \le H \cdot \int_{-\infty}^{+\infty} &\exp\{[\delta^2/(\beta p - (p/p')\cdot(\delta^2/\varepsilon)) - \varepsilon]y^2 \\ &+ [\gamma + (\delta\cdot(\xi p + (p/p')\cdot(\gamma\delta/\varepsilon)))/(\beta p - (p/p') \\ &\cdot(\delta^2/\varepsilon))]y\}\cdot|f(y)|^p\,dy,\end{aligned}$$

where

$$\begin{aligned}H = {} &(\pi/\varepsilon)^{p/2p'}\cdot(\pi/(\beta p - (p/p')\cdot(\delta^2/\varepsilon)))^{1/2} \\ &\cdot\exp\{(p\gamma^2/4\varepsilon p') + ((\xi p + (p/p')\cdot(\gamma\delta/\varepsilon))^2)/(4\cdot(\beta p - (p/p')\cdot(\delta^2/\varepsilon)))\}.\end{aligned}$$

Proof. By writing

$$\begin{aligned}(\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(x) = \int_{-\infty}^{+\infty} &\{(\exp[-\beta x^2 - \varepsilon y^2 + 2\delta xy + \xi x + \gamma y])^{1/p}\cdot f(y)\} \\ &\cdot\{(\exp[-\beta x^2 - \varepsilon y^2 + 2\delta xy + \xi x + \gamma y])^{1/p'}\}\,dy,\end{aligned}$$

$x \in \mathbb{R}$, and applying Hölder's inequality, it follows that

$$|(\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(x)| \leq \left(\int_{-\infty}^{+\infty} \exp[-\beta x^2 - \varepsilon y^2 + 2\delta xy + \xi x + \gamma y] \cdot |f(y)|^p \, dy\right)^{1/p} \cdot \left(\int_{-\infty}^{+\infty} \exp[-\beta x^2 - \varepsilon y^2 + 2\delta xy + \xi x + \gamma y] \, dy\right)^{1/p'}.$$

Since for $\varepsilon > 0$,

$$\int_{-\infty}^{+\infty} \exp[-\beta x^2 - \varepsilon y^2 + 2\delta xy + \xi y + \gamma x] \, dy = (\pi/\varepsilon)^{1/2} \cdot \exp[((\delta^2/\varepsilon) - \beta)x^2 + (\xi + (\delta\gamma/\varepsilon))x + (\gamma^2/4\varepsilon)],$$

we arrive at the estimate

$$\begin{aligned} &|(\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f)(x)|^p \\ &\quad \leq (\pi/\varepsilon)^{p/2p'} \cdot \exp[p\gamma^2/4\varepsilon p'] \\ &\quad\quad \cdot \int_{-\infty}^{+\infty} \exp[-(\beta p - (p/p') \cdot (\delta^2/\varepsilon))x^2 - \varepsilon y^2 + 2\delta xy \\ &\quad\quad\quad + (\xi p + (p/p') \cdot (\gamma\delta/\varepsilon))x + \gamma y] \cdot |f(y)|^p \, dy. \end{aligned}$$

After integration with respect to x, the theorem follows. □

Corollary 3.1. *Under the same hypothesis and notation of Theorem* 3.1, *we set*

$$A = \delta^2/(\beta p - (p/p') \cdot (\delta^2/\varepsilon)) - \varepsilon$$

and

$$B = \gamma + (\delta \cdot (\xi p + (p/p') \cdot (\gamma\delta/\varepsilon)))/(\beta p - (p/p') \cdot (\delta^2/\varepsilon)).$$

Then,

(a) *If* $A = 0$ (*or equivalently,* $\delta^2 = \beta \cdot \varepsilon$) *and* $B = 0$, *one has*

$$\|\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_p \leq H^{1/p} \cdot \|f\|_p \quad \textit{for all } f \in L^p(\mathbb{R}).$$

Note that if $H \leq 1$, *we obtain*

$$\|\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_p \leq \|f\|_p \quad \textit{for all } f \in L^p(\mathbb{R}).$$

(b) *If* $A < 0$ (*or equivalently,* $\delta^2 < \beta \cdot \varepsilon$), *then*

$$\|\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_p \leq (H \cdot \exp(B^2/4A))^{1/p} \cdot \|f\|_p \quad \textit{for all } f \in L^p(\mathbb{R}).$$

Note that if $H \cdot \exp(B^2/4A) \leq 1$, *we obtain*

$$\|\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_p \leq \|f\|_p \quad \textit{for all } f \in L^p(\mathbb{R}).$$

Remark. For $1 < p < \infty$, $\varepsilon > 0$, $\delta \neq 0$, and $\delta^2 = \beta\varepsilon$, the question of contractivity remains open if $B \neq 0$ or $H > 1$. This question also remains open for $1 < p < \infty$, $\varepsilon > 0$, and $\delta^2 < \beta\varepsilon$, if $H \cdot \exp(B^2/4A) > 1$.

Theorem 3.2. *Let* $0 < p < \infty$, *and assume* $\varepsilon > 0$ *and* $\delta^2 < \beta\varepsilon$. *Then, for all* $f \in L^2(\mathbb{R})$,

$$\|\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma}f\|_p \leq H^* \cdot \|f\|_2, \tag{3.2}$$

where

$$H^* = (2\pi)^{1/4} \cdot (4\varepsilon)^{-1/4} \cdot (\pi/p)^{1/2p} \cdot (\beta - (\delta^2/\varepsilon))^{-1/2p} \cdot \exp\{(\gamma^2/4\varepsilon) + (p \cdot (\xi + (\delta\gamma/\varepsilon))^2)/(4 \cdot (\beta - (\delta^2/\varepsilon)))\}.$$

Proof. By Schwarz's inequality and the evaluation of a Gaussian integral, we obtain for $f \in L^2(\mathbb{R})$,

$$|(\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f)(x)| \leq (2\pi)^{1/4} \cdot (4\varepsilon)^{-1/4} \cdot \exp[((\delta^2/\varepsilon) - \beta)x^2 + (\xi + (\delta\gamma/\varepsilon))x + (\gamma^2/4\varepsilon)] \cdot \|f\|_2.$$

Again, by evaluating a Gaussian integral (3.2) follows. □

Corollary 3.2. *Under the same conditions and notation of Theorem* 3.2, *if* $H^* \leq 1$, *then*

$$\|\mathscr{F}_{\beta,\varepsilon,\delta,\xi,\gamma} f\|_p \leq \|f\|_2 \quad \textit{for all } f \in L^2(\mathbb{R}).$$

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