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FRAME PERTURBATIONS

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ABSTRACT. We consider the stability of Hilbert space frames under perturbations. Our results are in spirit close to classical results for orthonormal bases, due to Paley and Wiener.

A frame can be viewed as a "generalized orthonormal basis"; if $\{f_i\}_{i\in I}$ is a frame for the Hilbert space $\mathscr{H}$, then any $f \in \mathscr{H}$ can be written as an infinite linear combination of the elements f_i. The coefficients do not need to be unique, and in general the expansion is nonorthogonal. But frames are a much more flexible tool than orthonormal bases, and they play a big role in wavelet theory.

It is a classical result that a sufficiently small perturbation of an orthonormal basis gives a Riesz basis. Our aim here is to consider the similar problem for frames. Our approach is motivated by the book [Y] and a result in [H] about perturbations of atoms in Banach spaces.

Let $\mathscr{H}$ be a separable Hilbert space, with the inner product $\langle \cdot , \cdot \rangle$ linear in the first entry.

A family $\{f_i\}_{i\in I}$ of elements in $\mathscr{H}$ is called a *Bessel sequence* if

$$\exists B > 0 \colon \sum_{i\in I} |\langle f , f_i\rangle|^2 \leq B\|f\|^2 \quad \forall f \in \mathscr{H}. \tag{1}$$

If $\{f_i\}_{i\in I}$ is a Bessel sequence, then $\sum_{i\in I} c_i f_i$ converges unconditionally for all $\{c_i\} \in l^2(I)$ and the mapping $T\colon \{c_i\} \mapsto \sum_{i\in I} c_i f_i$ is bounded from $l^2(I)$ into $\mathscr{H}$, with $\|T\| \leq \sqrt{B}$. Composing T with the adjoint operator $T^*\colon f \mapsto \{\langle f , f_i\rangle\}_{i\in I}$ we get the *frame operator*

$$S\colon \mathscr{H} \to \mathscr{H}, \qquad Sf = \sum_{i\in I} \langle f , f_i\rangle f_i .$$

The Bessel sequence $\{f_i\}_{i\in I}$ is called a *frame* if

$$\exists A > 0 \colon A\|f\|^2 \leq \sum_{i\in I} |\langle f , f_i\rangle|^2 \quad \forall f \in \mathscr{H}. \tag{2}$$

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Any pair of numbers A and B such that (1) resp. (2) are satisfied will be called a *set of frame bounds*. The smallest possible upper bound is

$$\sup_{\|f\|=1} \sum_{i \in I} |\langle f, f_i \rangle|^2 = \sup_{\|f\|=1} |\langle Sf, f \rangle| = \|S\| = \|T\|^2.$$

If $\{f_i\}_{i \in I}$ is a frame, then S has a bounded inverse, defined on all of $\mathscr{H}$; this fact leads to the important *frame decomposition*

$$f = SS^{-1}f = \sum_{i \in I} \langle S^{-1}f, f_i \rangle f_i = \sum_{i \in I} \langle f, S^{-1}f_i \rangle f_i \quad \forall f \in \mathscr{H}.$$

$\{S^{-1}f_i\}_{i \in I}$ is also a frame, usually called the *dual frame*; as bounds one can use $\frac{1}{B}$ and $\frac{1}{A}$.

Theorem 1. *Let $\{f_i\}_{i \in I}$ be a frame for $\mathscr{H}$, with bounds A and B. Any family $\{g_i\}_{i \in I}$ of elements in $\mathscr{H}$ such that*

$$R := \sum_{i \in I} \|f_i - g_i\|^2 < A$$

is a frame for $\mathscr{H}$ with bounds $A(1 - \sqrt{\frac{R}{A}})^2$ and $B(1 + \sqrt{\frac{R}{B}})^2$.

Proof. Denote the frame operator for $\{f_i\}_{i \in I}$ by S. The assumptions imply that $\{g_i\}_{i \in I}$ is a Bessel sequence, so we can define a bounded linear operator

$$U: \mathscr{H} \to \mathscr{H}, \qquad Uf := \sum_{i \in I} \langle f, S^{-1}f_i \rangle g_i.$$

Now,

$$\begin{aligned} \|f - Uf\|^2 &= \left\| \sum_{i \in I} \langle f, S^{-1}f_i \rangle f_i - \langle f, S^{-1}f_i \rangle g_i \right\|^2 \\ &\leq \sum_{i \in I} |\langle f, S^{-1}f_i \rangle|^2 \cdot \sum_{i \in I} \|f_i - g_i\|^2 \leq \frac{R}{A} \|f\|^2 \quad \forall f \in \mathscr{H}. \end{aligned}$$

That is, $\|I - U\| \leq \sqrt{\frac{R}{A}} < 1$. So U is invertible, and

$$\|U\| \leq 1 + \sqrt{\frac{R}{A}}, \qquad \|U^{-1}\| \leq \frac{1}{1 - \sqrt{\frac{R}{A}}}.$$

Any $f \in \mathscr{H}$ can be written as

$$f = UU^{-1}f = \sum_{i \in I} \langle U^{-1}f, S^{-1}f_i \rangle g_i;$$

thus

$$\begin{aligned} \|f\|^4 &= \left| \left\langle \sum_{i \in I} \langle U^{-1}f, S^{-1}f_i \rangle g_i, f \right\rangle \right|^2 = \left| \sum_{i \in I} \langle U^{-1}f, S^{-1}f_i \rangle \langle g_i, f \rangle \right|^2 \\ &\leq \sum_{i \in I} |\langle U^{-1}f, S^{-1}f_i \rangle|^2 \cdot \sum_{i \in I} |\langle g_i, f \rangle|^2 \\ &\leq \frac{1}{A} \|U^{-1}f\|^2 \sum_{i \in I} |\langle g_i, f \rangle|^2 \leq \frac{\|f\|^2}{A(1 - \sqrt{\frac{R}{A}})^2} \cdot \sum_{i \in I} |\langle g_i, f \rangle|^2 \quad \forall f \in \mathscr{H}. \end{aligned}$$

So

$$A\left(1-\sqrt{\frac{R}{A}}\right)^2 \|f\|^2 \le \sum_{i\in I} |\langle g_i, f\rangle|^2 \quad \forall f \in \mathcal{H}.$$

Now define

$$T\colon l^2(I) \to \mathcal{H}, \qquad T\{c_i\} := \sum_{i\in I} c_i g_i.$$

The frame operator for $\{g_i\}_{i\in I}$ is TT^*, so the optimal upper frame bound for $\{g_i\}_{i\in I}$ is $\|T\|^2$. For $\{c_i\} \in l^2(I)$ we have

$$\|T\{c_i\}\| = \left\|\sum_{i\in I} c_i g_i\right\| \le \left\|\sum_{i\in I} c_i(g_i - f_i)\right\| + \left\|\sum_{i\in I} c_i f_i\right\| \le (\sqrt{B}+\sqrt{R})\|\{c_i\}\|.$$

So

$$\|T\|^2 \le (\sqrt{B}+\sqrt{R})^2 = B\left(1+\sqrt{\frac{R}{B}}\right)^2.$$

In some sense, the result is the best possible; if $\sum_{i\in I}\|f_i - g_i\|^2 = A$, then $\{g_i\}_{i\in I}$ does not even need to be total in $\mathcal{H}$. For example, if $\{f_i\}_{i=1}^\infty$ is an ONB, then $\{f_i\}_{i=1}^\infty$ is a frame with $A = B = 1$. If we define $g_1 = 0$, $g_i = f_i$, $i \ge 2$, then $\sum_{i\in I}\|f_i - g_i\|^2 = 1$ and $\{g_i\}$ is not total.

Lemma 2. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$. If $J \subseteq I$ is finite, then $\{f_i\}_{i\in I-J}$ is a frame for $\overline{\text{span}}\{f_i\}_{i\in I-J}$.*

Proof. Let $j \in I$; it is enough to prove that $\{f_i\}_{i\neq j}$ is a frame for $\overline{\text{span}}\{f_i\}_{i\neq j}$. Let P denote the orthogonal projection on $\overline{\text{span}}\{f_i\}_{i\neq j}$. Then $\{f_i\}_{i\neq j} \cup \{Pf_j\}$ is a frame for $\overline{\text{span}}\{f_i\}_{i\neq j}$. But $\{f_i\}_{i\neq j}$ is total in $\overline{\text{span}}\{f_i\}_{i\neq j}$ and therefore itself a frame for $\overline{\text{span}}\{f_i\}_{i\neq j}$; cf. [DS, Lemma 9].

Two families $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are said to be *quadratically close* if

$$\sum_{i\in I}\|f_i - g_i\|^2 < \infty.$$

Theorem 3. *Let $\{f_i\}_{i\in I}$ be a frame and $\{g_i\}_{i\in I}$ a family which is quadratically close to $\{f_i\}_{i\in I}$. Then $\{g_i\}_{i\in I}$ is a frame for $\overline{\text{span}}\{g_i\}_{i\in I}$.*

Proof. Again let A denote a lower frame bound for $\{f_i\}_{i\in I}$. There exists a finite index set $J \subseteq I$ such that $\sum_{i\in I-J}\|f_i - g_i\|^2 < A$. By Theorem 1 $\{f_i\}_{i\in J} \cup \{g_i\}_{i\in I-J}$ is a frame for $\mathcal{H}$. Now Lemma 2 shows that $\{g_i\}_{i\in I-J}$ is a frame for $\overline{\text{span}}\{g_i\}_{i\in I-J}$.

$\{g_i\}_{i\in I}$ is a Bessel sequence in $\mathcal{H}$. Observe that

$$\overline{\text{span}}\{g_i\}_{i\in I} = \overline{\text{span}}\{g_i\}_{i\in I-J} + \text{span}\{g_i\}_{i\in J};$$

it follows that the operator

$$T\colon l^2(I) \to \overline{\text{span}}\{g_i\}_{i\in I}, \qquad T\{c_i\} := \sum_{i\in I} c_i g_i$$

is surjective. Now [C, Corollary 4.2] implies that $\{g_i\}_{i\in I}$ is a frame for $\overline{\text{span}}\{g_i\}_{i\in I}$.

Remark. Let G be a topological group and π a strongly continuous unitary representation of G on the Hilbert space $\mathcal{H}$. In wavelet analysis one is interested in coherent frames, i.e., frames of the form $\{\pi(x_i)\}_{i\in I}$, where $\{x_i\}_{i\in I}$ is a set of group elements and $f \in \mathscr{H}$ (see [D, DGM, HW]). Our results cannot be applied to perturbations of f, since $\|\pi(x_i)f - \pi(x_i)g\| = \|f - g\|$. But if $\{y_i\}_{i\in I}$ is another family of group elements, then $\|\pi(x_i)f - \pi(y_i)f\| = \|f - \pi(x_i^{-1}y_i)f\|$. So our results show that $\{\pi(y_i)f\}_{i\in I}$ is a frame if $\{y_i\}_{i\in I}$ is sufficiently close to $\{x_i\}_{i\in I}$. We shall not go into details with concrete calculations here.

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