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CESÀRO MEANS OF FOURIER SERIES ON ROTATION GROUPS

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ABSTRACT. We study the Cesàro means of Fourier series on rotation groups SO(3) and SO(4). On these two classical groups, we solve an open question recently posted in *Harmonic analysis on classical groups* [Springer-Verlag, Berlin, and Science Press, Beijing, 1991].

Let $SO(n)$ be the rotation group on $\mathbb{R}^n$. By [H], it is known that one can identify $SO(n)$ as the characteristic manifold of the classical domain $\mathscr{R}_n$ which was studied by E. Cartan (see [C] for the definition of $\mathscr{R}_n$). To solve the Dirichlet problem on $\mathscr{R}_n$, Hua proved, from the view of several complex variables, that the Poisson kernel on $\mathscr{R}_n$ is (see [H])

$$P(X_0, \Gamma) = \frac{\det(I - X_0 X_0')^{(n-1)/2}}{\det(I - X_0\Gamma')^{n-1}}, \tag{1}$$

where I is the identity element in $SO(n)$ and X' is the transpose of a matrix X.

From the above explicit formula of the Poisson kernel, Gong defined the Cesàro kernel on $SO(n)$ as follows (see [G, p. 140]).

Let dV be the normalized Haar measure of $SO(n)$. For $\alpha > -1$ and any positive integer N, let $A_N^\alpha = (\alpha + N)\cdots(\alpha+1)/N!$. Then the Cesàro kernel $K_N^\alpha(V)$ on $SO(n)$ is defined by

$$K_N^\alpha(V) = \det{}^{(n-1)/2}\left(\left\{A_N^\alpha I + \sum_{j=1}^{N}(V^j + V'^j)\sum_{\nu=0}^{N-j} A_\nu^{\alpha-1}\right\}/A_N^\alpha\right)/B_N^\alpha, \tag{2}$$

where

$$B_N^\alpha = \int_{SO(n)} \det{}^{(n-1)/2}\left(\left\{A_N^\alpha I + \sum_{j=1}^{N}(V^j + V'^j)\sum_{\nu=0}^{N-j} A_\nu^{\alpha-1}\right\}/A_N^\alpha\right) dV. \tag{3}$$

We easily see, from the above definition, that for any integer N and any $V \in SO(n)$,

$$\int_{SO(n)} K_N^\alpha(UV)\,dU = 1. \tag{4}$$

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Because SO(2) can be identified with the one-dimensional torus $\mathbb{T}$, it is also easy to see that the above definition (2) is the classical (c, α) kernel of the Fourier series on $\mathbb{T}$ (see [Z]) when $n = 2$.

In [G], Gong proved the following convergence theorem on $SO(n)$:

Theorem A. *Suppose that f is any continuous function on $SO(n)$. If the index α is greater than $(n-2)/(n-1)$, then*

$$\lim_{N\to\infty}(K_N^\alpha * f)(V) = f(V). \tag{5}$$

It is a well-known fact that (see [Z]) in Theorem A the condition $\alpha > (n-2)/(n-1)$ is sharp for $n = 2$. Thus, Gong posed an open question: whether the condition $\alpha > (n-2)/(n-1)$ can be improved when $n \geq 3$?

In this note, we solve the above question on SO(3) and SO(4). The main result consists of the following two theorems.

Theorem 1 (Result on SO(3)). (i) *Suppose that f is a continuous function on* SO(3). *If $\alpha_0 = \frac{1}{2}$, then for any $V \in SO(3)$, $\lim_{N\to\infty}(K_N^{\alpha_0} * f)(V) = f(V)$.*

(ii) *For any $\alpha \in (-1, \frac{1}{2})$ there is a C^∞ function $g(V)$ on* SO(3) *such that*

$$\overline{\lim_{N\to\infty}}(K_N^\alpha * g)(I) \neq g(I).$$

Theorem 2 (Result on SO(4)). (i) *Let $\alpha_0 = \frac{2}{3}$; then*

$$\int_{SO(4)} |K_N^{\alpha_0}(V)| dV \geq A \log N \quad \text{as } N \to \infty.$$

(ii) *For $\alpha \in (-1, 0)$,*

$$\int_{SO(4)} |K_N^{\alpha}(V)| dV \geq AN \quad \text{as } N \to \infty.$$

(iii) *For $\alpha \in (\frac{1}{2}, \frac{2}{3})$,*

$$\int_{SO(4)} |K_N^{\alpha}(V)| dV \geq AN^{2-3\alpha} \quad \text{as } N \to \infty.$$

(iv) *For $\alpha \in [0, \frac{1}{2})$,*

$$\int_{SO(4)} |K_N^{\alpha}(V)| dV \geq AN^{1-\alpha} \quad \text{as } N \to \infty.$$

(v)

$$\int_{SO(4)} |K_N^{1/2}(V)| dV \geq AN^{1/2}/\log N \quad \text{as } N \to \infty.$$

In the above formulas, A is a constant independent of N.

Notes. By the well-known Banach-Steinhaus theorem, Theorem 2 implies that Theorem A fails on SO(4) if $\alpha \in (-1, \frac{2}{3}]$.

Before proving these two theorems, we need to derive a more explicit formula of the kernel $K_N^\alpha(V)$.

Let $S(\theta)$ be the 2×2 matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and $C(\theta)$ be the 2×2 matrix

$$\begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix}.$$

Recall that any $V \in \mathrm{SO}(2k)$ is conjugate to a $2k \times 2k$ matrix $T(\theta)$ which is $S(\theta_1) \oplus S(\theta_2) \oplus \cdots \oplus S(\theta_k)$ and that any $V \in \mathrm{SO}(2k+1)$ is conjugate to a $(2k+1) \times (2k+1)$ matrix $T(\theta)$ which equals $S(\theta_1) \oplus S(\theta_2) \oplus \cdots \oplus S(\theta_k) \oplus 1$, where $(\theta_1, \theta_2, \dots, \theta_k)$ is a coordinate satisfying

$$-\pi \le \theta_j \le \pi, \qquad j = 1, 2, \dots, k.$$

Noticing that $\sum_{\nu=0}^{N-j} A_\nu^{\alpha-1} = A_{n-j}^\alpha$ (see [Z, p. 77]) and that the determinant is a central function, we easily see that

$$K_N^\alpha(V) = \det{}^{(n-1)/2}\left(\left\{A_N^\alpha I + \sum_{j=1}^N (T(\theta)^j + T(\theta)'^j) A_{N-j}^\alpha\right\} / A_N\right) / B_N^\alpha.$$

Using induction, we also easily obtain that

$$T(\theta)^j = T(j\theta) \quad \text{and} \quad T(\theta)'^j = T(j\theta)'. \tag{6}$$

Thus by the definition of $T(\theta)$, if $n = 2k$, then

$$T(\theta)^j + T(\theta)'^j = 2^k C(j\theta_1) \oplus C(j\theta_2) \oplus \cdots \oplus C(j\theta_k). \tag{7}$$

In this case we obtain that

$$\left\{A_N^\alpha I + \sum_{j=1}^N (T(\theta)^j + T(\theta)'^j) A_{N-j}^\alpha\right\} / A_N^\alpha$$

is a $2k \times 2k$ matrix:

$$\sigma_N^\alpha(\theta_1) \oplus \sigma_N^\alpha(\theta_1) \oplus \sigma_N^\alpha(\theta_2) \oplus \sigma_N^\alpha(\theta_2) \oplus \cdots \oplus \sigma_N^\alpha(\theta_k) \oplus \sigma_N^\alpha(\theta_k),$$

where $\sigma_N^\alpha(\theta) = \frac{1}{2} + \sum_{j=1}^N \cos j\theta A_{N-j}^\alpha / A_N^\alpha$ is the one-dimensional Cesàro kernel (see [Z, 1.14, p. 77 and 5.2, p. 94]). Therefore, by noticing the definition of B_N^α, we clearly see that the Cesàro kernel on $\mathrm{SO}(2k)$ is

$$K_N^\alpha(V) = \prod_{j=1}^k \{\sigma_N^\alpha(\theta_j)\}^{2k-1} / \tilde{B}_N^\alpha, \tag{8}$$

where

$$\tilde{B}_N^\alpha = \int \cdots \int_{-\pi \le \theta_k \le \cdots \le \theta_1 \le \pi} \prod_{j=1}^k \{\sigma_N^\alpha(\theta_j)\}^{2k-1} \prod_{1 \le i < j \le k} (\cos\theta_i - \cos\theta_j)^2 d\theta_1 \cdots d\theta_k$$

and $\prod_{1 \le i < j \le k} (\cos\theta_i - \cos\theta_j)^2$ is the Weyl function, up to a constant multiplier, on $\mathrm{SO}(2k)$ (see [W]).

If $n = 2k + 1$, then

$$T(\theta)^j + T(\theta)'^j = 2^{k+1} C(j\theta_1) \oplus C(j\theta_2) \oplus \cdots \oplus C(j\theta_k) \oplus 1. \tag{9}$$

Similar to the case of $n = 2k$, we easily see that

$$\begin{aligned} A_N^\alpha I + \sum_{j=1}^{N}(T(\theta)^j + T(\theta)'^j)A_{N-j}^\alpha / A_N^\alpha \\ = \sigma_N^\alpha(\theta_1) \oplus \sigma_N^\alpha(\theta_1) \oplus \cdots \oplus \sigma_N^\alpha(\theta_k) \oplus \sigma_N^\alpha(\theta_k) \oplus (2N+1). \end{aligned}$$

Thus the Cesàro kernel on $SO(2k+1)$ is

$$K_N^\alpha(V) = \prod_{j=1}^{k}\{\sigma_N^\alpha(\theta_j)\}^{2k} / \tilde{B}_N^\alpha, \tag{10}$$

where

$$\begin{aligned} \tilde{B}_N^\alpha = \int \cdots \int_{-\pi \le \theta_k \le \cdots \le \theta_1 \le \pi} \prod_{j=1}^{k}\{\sigma_N^\alpha(\theta_j)\}^{2k} \\ \times (1 - \cos\theta_j) \prod_{1 \le i < j \le k} (\cos\theta_i - \cos\theta_j)^2 d\theta_1 \cdots d\theta_k \end{aligned}$$

and

$$\prod_{j=1}^{k}(1 - \cos\theta_j) \prod_{1 \le i < j \le k} (\cos\theta_i - \cos\theta_j)$$

is the Weyl function, up to a constant multiplier, on $SO(2k+1)$ (see [W]).

Recall the following estimates of the classical Cesàro kernels:

Lemma 1. *If $\alpha > -1$ and $|\theta| \le N^{-1}$, then $BN \ge \sigma_N^\alpha(\theta) \ge AN$, where A, B are positive constants independent of θ and N. If $\alpha > -1$ and $|\theta| > N^{-1}$, then*

$$\begin{aligned} \{\sigma_N^\alpha(\theta)\}^n = \sin^n\{(N + (1+\alpha)/2)\theta - \pi\alpha/2\} / (\alpha)_N^n (2\sin(\theta/2))^{n(\alpha+1)}\} \\ + O(N^{-(n-1)\alpha - 1}\theta^{-(n-1)(\alpha+1)-2}), \end{aligned} \tag{11}$$

where $(\alpha)_N = \Gamma(\alpha + N + 1) / \{\Gamma(\alpha+1)\Gamma(N+1)\} \cong N^\alpha$ for N sufficiently large.

Proof. See [Z, pp. 77, 95] for the proof. □

Now we are ready to prove the main theorems.

Proof of Theorem 1. Let $d(U, I)$ be the Euclidean distance between U and I; then $d(U, I) = 2^{1/2}(1 - \cos\theta)^{1/2}$, where U is conjugate to the element $S(\theta) \oplus 1$ (see [G, p. 153]). We denote the modulus of continuity of a continuous function f by $\omega(f; \delta)$. Then by noticing formula (4) and that $K_N^\alpha(V)$ is a positive kernel on $SO(2k+1)$, we have

$$\begin{aligned} &|(K_N^{\alpha_0} * f)(V) - f(V)| \\ &= \left| \int_{SO(3)} K_N^{\alpha_0}(U)\{f(U^{-1}V) - f(V)\} dU \right| \\ &\le \omega(f; 2\delta) \int_{|\theta| \le \delta} K_N^{\alpha_0}(U) dU + 2\|f\|_\infty \int_{|\theta| \ge \delta} K_N^{\alpha_0}(U) dU \\ &\le \omega(f; 2\delta) + 2\|f\|_\infty \int_{|\theta| \ge \delta} K_N^{\alpha_0}(U) dU. \end{aligned}$$

Notice that $\omega(f;2\delta)$ goes to zero as δ tends to zero. Thus to prove (i) in Theorem 1, it suffices to show that for any fixed $\delta > 0$,

$$\lim_{N\to\infty}\int_{|\theta|\geq\delta} K_N^{\alpha_0}(U)dU = 0. \tag{12}$$

By (10), we know that

$$\int_{|\theta|\geq\delta} K_N^{\alpha_0}(U)dU = \int_{|\theta|\geq\delta}\{\sigma_N^{\alpha_0}(\theta)\}^2(1-\cos\theta)d\theta/\widetilde{B}_N^{\alpha_0}, \tag{13}$$

where

$$\widetilde{B}_N^{\alpha_0} = \int_{-\pi}^{\pi}\{\sigma_N^{\alpha_0}(\theta)\}^2(1-\cos\theta)d\theta.$$

By the definition of the Cesàro kernel together with (11), one easily sees that

$$\begin{aligned}&\int_{|\theta|\geq\delta}\{\sigma_N^{\alpha_0}(\theta)\}^2(1-\cos\theta)d\theta \\ &\qquad\leq N^{-1}\int_{|\theta|\geq\delta}|\theta|^{-3}(1-\cos\theta)d\theta \leq A_\delta N^{-1},\end{aligned} \tag{14}$$

where A_δ is a constant depending only on δ.

On the other side, by (11) we have

$$\begin{aligned}\widetilde{B}_N^{\alpha_0} &\geq N^{-1}\int_{1/N}^{\pi}\sin^2\{(N+\tfrac{3}{4})\theta-\pi/4\}\sin^{-3}(\theta/2)(1-\cos\theta)d\theta \\ &\quad + O\left(N^{-3/2}\int_{1/N}^{\pi}\theta^{-7/2}(1-\cos\theta)d\theta\right) \\ &\geq N^{-1}\int_{1/N}^{\pi}\sin^2\{(N+\tfrac{3}{4})\theta-\pi/4\}\theta^{-1}d\theta + O(N^{-1}).\end{aligned}$$

This shows that

$$\widetilde{B}_N^{\alpha_0} \geq AN^{-1}\log N \qquad (N\to\infty). \tag{15}$$

Equations (14) and (15) furnish the proof of (i) in Theorem 1.

Next let $g(U) = (1-\cos\theta)$; then $g(I) = 0$. We want to prove that this C^∞ function $g(U)$ furnishes the second part of Theorem 1. In fact,

$$(K_N^\alpha * g)(I) - g(I) = \int_{SO(3)} K_N^\alpha(U)g(U)dU = I_N^\alpha/\widetilde{B}_N^\alpha,$$

where

$$I_N^\alpha = \int_{-\pi}^{\pi}\{\sigma_N^\alpha(\theta)\}^2(1-\cos\theta)^2d\theta,$$

$$\widetilde{B}_N^\alpha = \int_{-\pi}^{\pi}\{\sigma_N^\alpha(\theta)\}^2(1-\cos\theta)d\theta.$$

Noticing that $\alpha\in(-1,\tfrac{1}{2})$, by Lemma 1 we have

$$\widetilde{B}_N^\alpha = O(1/N) + O\left(N^{-2\alpha}\int_{1/N}^{\pi}\theta^{-2(\alpha+1)+2}d\theta\right) = O(N^{-2\alpha}) \quad \text{as } N\to\infty. \tag{16}$$

Using formula (11) and noticing $\alpha \in (-1, \frac{1}{2})$, we obtain that

$$I_N^\alpha \geq AN^{-2\alpha} \int_{1/N}^{\pi} \sin^2\{(N + (1+\alpha)/2)\theta - \pi\alpha/2)\}(1-\cos\theta)^2 \sin^{-2(\alpha+1)}(\theta/2)d\theta$$
$$+ O\left(N^{-\alpha-1} \int_{1/N}^{\pi} \theta^{-\alpha-3}(1-\cos\theta)^2 d\theta\right).$$

Thus, an easy computation shows that

$$I_N^\alpha \geq AN^{-2\alpha} \qquad (N \to \infty). \tag{17}$$

From (16) and (17), we know that

$$\overline{\lim_{N\to\infty}} \int_{SO(3)} K_N^\alpha(U)g(U)dU > 0 = g(I).$$

Theorem 1 is proved. □

Proof of Theorem 2. Let $\alpha \in (-1, \frac{2}{3}]$. We need to calculate the Lebesgue constant $K_N^\alpha(V)$. By formula (8), we know that $\int_{SO(4)} |K_N^\alpha(V)|dV$ is equal to

$$\iint\limits_{-\pi\leq\theta_2<\theta_1\leq\pi} |\sigma_N^\alpha(\theta_1)\sigma_N^\alpha(\theta_2)|^3(\cos\theta_1 - \cos\theta_2)^2 d\theta_1 d\theta_2/\widetilde{B}_N^\alpha = J_N^\alpha/\widetilde{B}_N^\alpha. \tag{18}$$

By a symmetric argument, $\widetilde{B}_N^\alpha$ in (18) is equal to

$$\begin{aligned} &2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\prod_{k=1}^{2}\{\sigma_N^\alpha(\theta_k)\}^3\{(1-\cos\theta_1)-(1-\cos\theta_2)\}^2 d\theta_1\, d\theta_2 \\ &= 2\left\{\iint\limits_{\substack{|\theta_1|\leq1/N\\|\theta_2|\leq1/N}} + \iint\limits_{\substack{|\theta_1|\geq1/N\\|\theta_2|\geq1/N}} + 2\iint\limits_{\substack{|\theta_1|\geq1/N\\|\theta_2|\leq1/N}}\right\} = J + JJ + JJJ. \end{aligned} \tag{19}$$

One easily sees that for any $\alpha > -1$,

$$J = O(1) \qquad (N \to \infty). \tag{20}$$

By (11) again, the third term JJJ in (19) is dominated by

$$O\left(N^{3-3\alpha}\int_{-1/N}^{1/N}\int_{|\theta_1|\geq1/N} \sin^3\{(N+(1+\alpha)/2)\theta_1 - \pi\alpha/2\}\right.$$
$$\left. \times\{|\theta_1|^{-3(\alpha+1)}\theta_2^4 + |\theta_1|^{-3\alpha-1}\theta_2^2 + |\theta_1|^{-3\alpha+1}\}\, d\theta_1\, d\theta_2\right)$$
$$+ O\left(N^{2-2\alpha}\int_{-1/N}^{1/N}\int_{|\theta_1|\geq1/N}\{|\theta_1|^{-2\alpha-4}\theta_2^4 + |\theta_1|^{-2\alpha-2}\theta_2^2 + |\theta_1|^{-2\alpha}\}d\theta_1 d\theta_2\right)$$
$$= JJJ(1) + JJJ(2).$$

It is easy to calculate that

$$JJJ(2) = \begin{cases} O(\log N) & \text{if}\, \alpha = \frac{1}{2}, \\ O(N^{1-2\alpha}) & \text{if}\, \alpha < \frac{1}{2}, \\ O(1) & \text{if}\, \alpha > \frac{1}{2}. \end{cases} \tag{21}$$

To estimate $JJJ(1)$, we need the following formula which can easily be proved by integration by parts:

$$\int_{1/N}^{\pi} \sin^3(N\theta - \pi\alpha/2)\,\theta^{\nu}\,d\theta = \begin{cases} O(N^{-1}) & \text{if } \nu \geq 0, \\ O(N^{-1-\nu}) & \text{if } \nu \in (-1, 0). \end{cases} \tag{22}$$

Obviously,

$$JJJ(1) = O\left(N^{2-3\alpha}\int_{1/N}^{\pi} \sin^3\{(N+(1+\alpha)/2)\theta_1 - \pi\alpha/2\}\,\theta_1^{1-3\alpha}d\theta_1\right).$$

So by (22), we have

$$JJJ(1) = O(N^{1-3\alpha}) \quad \text{if } \alpha \in (-1, \tfrac{1}{3}), \tag{23}$$

$$JJJ(1) = O(1) \quad \text{if } \alpha \in [\tfrac{1}{3}, \tfrac{2}{3}]. \tag{24}$$

Combining (21), (23), and (24), we have

$$JJJ = \begin{cases} O(\log N), & \alpha = \tfrac{1}{2}, \\ O(N^{1-2\alpha}), & \alpha \in [0, \tfrac{1}{2}), \\ O(1), & \alpha > \tfrac{1}{2}; \end{cases} \tag{25}$$

$$JJJ = O(N^{1-3\alpha}), \qquad \alpha \in (-1, 0). \tag{25'}$$

We now estimate the term JJ in (19).

Clearly,

$$\begin{aligned} JJ &= O\left(\int_{1/N}^{\pi}\{\sigma_N^{\alpha}(\theta)\}^3 d\theta_1 \int_{1/N}^{\pi}\{\sigma_N^{\alpha}(\theta_2)\}^3\theta_2^4 d\theta_2\right) \\ &\quad + O\left(\prod_{k=1}^{2}\int_{1/N}^{\pi}\{\sigma_N^{\alpha}(\theta_k)\}^3\theta_k^2 d\theta_k\right) \\ &= JJ(1) + JJ(2). \end{aligned}$$

Let $si(N, \theta, \alpha) = \sin\{(N+(1+\alpha)/2)\theta - \pi\alpha/2\}$. Using the same method used in estimating the term JJJ, we easily obtain

$$\begin{aligned} \int_{1/N}^{\pi}\{\sigma_N^{\alpha}(\theta)\}^3 d\theta &= O\left(N^{-3\alpha}\int_{1/N}^{\pi} si^3(N, \theta, \alpha)\,\theta^{-3(\alpha+1)}d\theta\right) \\ &\quad + O\left(N^{-2\alpha-1}\int_{1/N}^{\pi}\theta^{-2\alpha-4}d\theta\right) \\ &= O(N^2). \end{aligned} \tag{26}$$

$$\begin{aligned} \int_{1/N}^{\pi}\{\sigma_N^{\alpha}(\theta)\}^3\theta^4\,d\theta &= O\left(N^{-3\alpha}\int_{1/N}^{\pi} si^3(N, \theta, \alpha)\,\theta^{-3\alpha+1}\,d\theta\right) \\ &\quad + O\left(N^{-2\alpha-1}\int_{1/N}^{\pi}\theta^{-2\alpha}d\theta\right) \\ &= \begin{cases} O(N^{-2}\log N) & \text{if } \alpha = \tfrac{1}{2}, \\ O(N^{-2\alpha-1}) & \text{if } \alpha < \tfrac{1}{2}, \\ O(N^{-2}) & \text{if } \alpha > \tfrac{1}{2}. \end{cases} \end{aligned} \tag{27}$$

Equations (26) and (27) imply that

$$JJ(1) = \begin{cases} O(\log N) & \text{if } \alpha = \frac{1}{2}, \\ O(N^{1-2\alpha}) & \text{if } \alpha \in (-1, \frac{1}{2}), \\ O(1) & \text{if } \alpha \in (\frac{1}{2}, \frac{2}{3}). \end{cases} \tag{28}$$

Similarly,

$$\begin{aligned} \int_{1/N}^{\pi} \{\sigma_N^\alpha(\theta)\}^3 \theta^2 \, d\theta &= O\left(N^{-3\alpha} \int_{1/N}^{\pi} si^3(N, \theta, \alpha)\, \theta^{-3\alpha-1} \, d\theta\right) \\ &\quad + O\left(N^{-2\alpha-1} \int_{1/N}^{\pi} \theta^{-2\alpha-2} d\theta\right) \\ &= \begin{cases} O(1) & \text{if } \alpha \in [-\frac{1}{3}, \frac{2}{3}], \\ O(N^{-3\alpha-1}) & \text{if } \alpha \in (-1, -\frac{1}{3}). \end{cases} \end{aligned} \tag{29}$$

Thus,

$$JJ(2) = \begin{cases} O(1) & \text{if } \alpha \in (-\frac{1}{3}, \frac{2}{3}], \\ O(N^{-6\alpha-2}) & \text{if } \alpha \in (-1, -\frac{1}{3}]. \end{cases} \tag{30}$$

Combining (20), (25), (25′), (28), and (30), we obtain that

$$\tilde{B}_N^\alpha = \begin{cases} O(\log N) & \text{if } \alpha = \frac{1}{2}, \\ O(1) & \text{if } \alpha \in (\frac{1}{2}, \frac{2}{3}], \\ O(N^{1-2\alpha}) & \text{if } \alpha \in [0, \frac{1}{2}), \\ O(N^{1-3\alpha}) & \text{if } \alpha \in (-1, 0). \end{cases} \tag{31}$$

On the other hand we have

$$\begin{aligned} J_N^\alpha &\geq \iint_{0 \leq 2\theta_2 \leq 1/N \leq \theta_1 \leq \pi/2} \prod_{k=1}^{2} |\sigma_N^\alpha(\theta_k)|^3 \sin^4(\theta_1/2) \, d\theta \\ &\geq AN^2 \int_{1/N}^{\pi/2} |\sigma_N^\alpha(\theta_1)|^3 \sin^4(\theta_1/2) \, d\theta \\ &\geq AN^{2-3\alpha} \int_{1/N}^{\pi/2} |\sin^3\{(N + (\alpha+1)/2)\theta - \alpha\pi/2\}| \sin^{-3\alpha+1} \theta \, d\theta \\ &\quad + O\left(N^{-2\alpha+1} \int_{1/N}^{\pi/2} \theta^{-2\alpha} d\theta\right). \end{aligned} \tag{32}$$

Therefore, an easy computation shows that for sufficiently large N,

$$J_N^\alpha \geq \begin{cases} A \log N & \text{if } \alpha = \frac{2}{3}, \\ AN^{2-3\alpha} & \text{if } \alpha \in (-1, \frac{2}{3}). \end{cases} \tag{33}$$

Finally, from (31) and (33), we know that there is a positive constant A independent of N such that

$$\int_{\mathrm{SO}(4)} |K_N^\alpha(V)| dV \geq \begin{cases} A \log N & \text{if } \alpha = \frac{2}{3}, \\ AN^{2-3\alpha} & \text{if } \alpha \in (\frac{1}{2}, \frac{2}{3}), \\ AN^{1-\alpha} & \text{if } \alpha \in [0, \frac{1}{2}), \\ AN & \text{if } \alpha \in (-1, 0) \end{cases}$$

and

$$\int_{SO(4)} |K_N^{1/2}(V)|\, dV \geq AN^{1/2}/\log N.$$

Theorem 2 is now proved. □

Furthermore, we can obtain an almost everywhere convergence theorem on SO(3):

Theorem 3. *If f is a Lebesgue integrable function on* SO(3), *then*

$$\lim_{N\to\infty} (K_N^{1/2} * f)(U) = f(U) \quad \textit{for almost all } U \in \mathrm{SO}(3).$$

Proof. Let $K^*f(U) = \sup_{N\geq 1} |(K_N^{1/2} * f)(U)|$, and let $\mathrm{Mf}(U)$ be the Hardy-Littlewood maximal function of f. If we can show $K^*f(U) \leq A\,\mathrm{Mf}(U)$ with A being a constant independent of f, then the theorem follows easily by a standard argument (see [SW] or [B]). Checking the proof of Theorem 1, we know that

$$K_N^{1/2} * f(U) = (\log N)^{-1} O\left(N \int_{SO(3)} \{\sigma_N^{1/2}(\theta)\}^2 f(UV) dV\right),$$

where V is conjugate to the element $S(\theta) \oplus 1$.

Let

$$\begin{aligned} &N \log^{-1} N \int_{SO(3)} \{\sigma_N^{1/2}(\theta)\}^2 f(UV)\, dV \\ &\quad = \log^{-1} N \left\{ \sum_{k=0}^{\log_2(N)} N \int_{2^k/N \leq d(V,I) \leq 2^{k+1}/N} + N \int_{0 \leq d(V,I) \leq 1/N} \right\} \\ &\quad = \log^{-1} N I(1) + \log^{-1} N I(2). \end{aligned}$$

It is easy to see that $(\log N)^{-1} |I(2)| \leq A\,\mathrm{Mf}(U)$.

By Lemma 1,

$$\begin{aligned} &\log^{-1} N |N \int_{2^k/N \leq d(V,I) \leq 2^{k+1}/N} \{\sigma_N^{1/2}(\theta)\}^2 f(UV)\, dV| \\ &\quad \leq \log^{-1} N \int_{2^k/N \leq d(V,I) \leq 2^{k+1}/N} |\theta|^{-3} |f(UV)|\, V \leq A \log^{-1} N\, \mathrm{Mf}(U). \end{aligned}$$

Therefore, $|K_N^{1/2} * f(U)| \leq A\,\mathrm{Mf}(U)$ with A being a constant independent of N. Theorem 3 is now proved. □

Recently we obtained some partial results on $\mathrm{SO}(n)$ for n being greater than four. In the higher-dimensional case, computations are much more complicated than those in the cases of $n = 3$ and $n = 4$. So though this paper is working with SO(3) and SO(4), it clearly demonstrates how to work on the higher-dimensional cases.

Finally we want to end this paper with a conjecture which is a well-known fact for $k = 1$:

Conjecture. Let $\alpha_0 = (2k-2)/(2k-1)$; then for large N

$$\int_{SO(2k)} |K_N^{\alpha_0}(V)|\, dV \cong \log N.$$

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