

**ASYMPTOTIC SOLUTION OF A SMALL PARAMETERED
 2-D INTEGRAL EQUATION ARISING FROM A CONTACT PROBLEM
 OF ELASTICITY BASED ON THE SOLUTION
 OF A 2-D INTEGRAL EQUATION**

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ABSTRACT. Asymptotic solution of a 2-D integral equation of constant kernel with small parameter ε ,

$$\int_0^\pi \int_{-\infty}^\infty p \, ds \, d\psi + \varepsilon r \int_0^\pi \int_{-\infty}^\infty p \, ds \, \cos \psi \, d\psi = G(r),$$

which occurs in a more exact form of Hertz's contact problem in elasticity, is presented in this paper based on the solution of a 2-D integral equation

$$\int_0^\pi \int_{-\infty}^\infty p \, ds \, d\psi = F(r)$$

with constant kernel, and the unknown function $p = p(s, \psi) = p(t, \phi)$ is subjected to the following two constraints:

$$\begin{aligned} p(t, \phi) &= p(t) \quad \forall \phi, \\ p(s, \psi) &= 0 \quad \text{for } (s, \psi) = (t, \phi) \notin E = \{(t, \phi) | t \leq a\} \end{aligned}$$

where (s, ψ) are local polar coordinates with origin at $M(r, 0)$, with $(r, 0)$ measured by global polar coordinates (t, ϕ) with origin at $O(0, 0)$. A more exact solution of Hertz's contact problem is found as an example.

1. INTRODUCTION

Contact problems have important applications in mechanical and civil engineering. Hertz's contact problem is a well-known classical problem in elasticity, and it is introduced by many textbooks (e.g., [1]). Hertz (1882) found, by conjecture, the solution of a contact problem of pressure distribution p between two elastic balls in contact, governed by the integral equation

$$(1) \quad \int_0^\pi \int_{-\infty}^\infty p \, ds \, d\psi = (\alpha - \beta r^2)/(k_1 + k_2),$$

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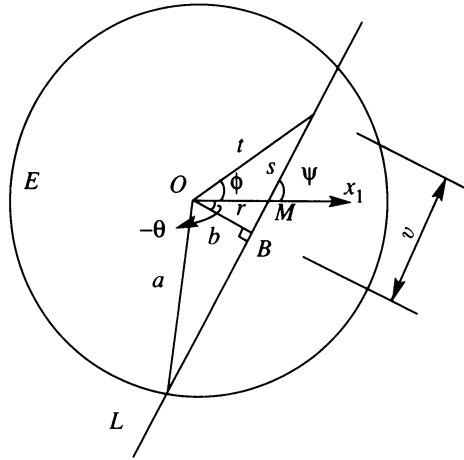


FIGURE 1

where $\alpha, \beta, k_1,$ and k_2 are constants, and the unknown pressure p is subjected to the constraints

$$(2) \quad p(t, \phi) = p(t) \quad \forall \phi,$$

$$(3) \quad p(s, \psi) = 0 \quad \text{for } (s, \psi) = (t, \phi) \notin E = \{(t, \phi) | t \leq a\},$$

where (s, ψ) are local polar coordinates with origin at $M(r, 0)$ with $(r, 0)$ measured by global polar coordinates (t, ϕ) with origin at $O(0, 0)$ (Figure 1). However, as pointed out in [2], the classical Hertz's integral equation (1) is not an exact equation for describing contact solids since the horizontal displacement is neglected there. An exact integral equation of Hertz's contact problem, where both vertical and horizontal components of displacement of points at contact surfaces due to pressure p are taken into consideration, was derived in [2] and is shown as

$$(4) \quad \iint p \, ds \, d\psi + \epsilon r \iint p \, ds \, \cos \psi \, d\psi + \delta \left(\iint p \, ds \, \cos \psi \, d\psi \right)^2 = (\alpha - \beta r^2)/(k_1 + k_2),$$

where the limits of integration are the same as (1) and are omitted for convenience, and $\alpha, \beta, k_1, k_2, \epsilon,$ and δ are constants. To conserve space, the expressions of these constants are omitted and we just mention that $0 < \delta \ll \epsilon \ll 1$. If the small high-order term δ is omitted and the right-hand side is replaced by a more general function $G(r)$, then (4) becomes

$$(5) \quad \int_0^\pi \int_{-\infty}^\infty p \, ds \, d\psi + \epsilon r \int_0^\pi \int_{-\infty}^\infty p \, ds \, \cos \psi \, d\psi = G(r).$$

If the small term ϵ is further neglected, then (5) can be written as

$$(6) \quad \int_0^\pi \int_{-\infty}^\infty p \, ds \, d\psi = F(r),$$

where $F(r)$ is a given function. If we take $F(r) = (\alpha - \beta r^2)/(k_1 + k_2)$, then (6) reduces to (1).

Many studies of differential equations with small parameters can be found, but studies of integral equations with small parameters are rarely seen. This paper studies the solution of the 2-D integral equation (5) with small parameter ε by asymptotic expansion based on the solution of the 2-D integral equation (6). Although Hertz found the solution of integral equation (1) by a conjecture, the solution of Hertz's integral equation (1) has not been derived in a strict mathematical way. In §2 the solution of the 2-D integral equation (6), which is more general than Hertz's integral equation (1), is derived by introducing the Radon transform, changing variables, and using the property of symmetry shown in (2). In §3 the integral equation (5) is solved by asymptotic expansion. Finally, as an example of the application of the asymptotic solution of (5), a comparison between the asymptotic solution and Hertz's solution is given, which shows that the classical Hertz solution is not exact enough for engineering applications but that the asymptotic solution is better in satisfying the criterion of the integral equation (5).

2. SOLUTION OF INTEGRAL EQUATION (6) SUBJECTED TO (2) AND (3)

No published paper concerned with integral equation (6) subjected to constraints (2) and (3) has been found. Here, we try to solve this problem as follows.

CT (computerized tomography) is a revolutionary medical instrument. Its appearance shocked the medical world, winning for its inventors the 1979 Nobel Medicine Prize. The Radon transform could be considered as a mathematical fundamental of image reconstruction from projections, especially for CT. Therefore, the Radon transform is an interesting and fairly hot topic at the moment. Here, we introduce the Radon transform to our problem. A Radon transform of function f of two variables (s, ψ) in local polar coordinates is also a function of two variables (b, θ) but in global polar coordinates and is defined [3] by

$$(7) \quad [Rf](b, \theta) \equiv \int_{-\infty}^{\infty} f(\sqrt{b^2 + v^2}, -\theta + \tan^{-1}(v/b)) dv.$$

The Radon transform of f represents a line integration of $f(s, \psi)$ ($= f(t, \phi)$) along a line L with angle ψ (Figure 1), where b is the distance between $O(0, 0)$ and L , θ is the angle between OB and the x_1 -axis, $OB \perp L$, and the function f satisfies that

$$(8) \quad f(s, \psi) = 0 \quad \text{for } (s, \psi) \notin E,$$

where E is the domain of function f .

The relationship between local polar coordinates (s, ψ) and global polar coordinates (t, ϕ) is

$$(9a) \quad s \cdot \sin \psi = t \cdot \sin \phi,$$

$$(9b) \quad s \cdot \cos \psi = t \cdot \cos \phi - r.$$

Now, according to the definition of the Radon transform of a function, the

left-hand side (L.H.S) of (6) can be written as

$$\begin{aligned} \text{L.H.S.} &= \int_0^\pi \int_{-\infty}^\infty p(s, \psi) ds d\psi = \int_0^\pi \int_{-\infty}^\infty p(t, \phi) ds d\psi \\ &= \int_0^\pi \int_{-\infty}^\infty p(\sqrt{v^2 + b^2}, (-\theta) + \tan^{-1}(v/b)) ds d\psi \\ &= \int_0^\pi [Rp](b, \theta) d\psi = \int_{-\pi/2}^{\pi/2} [Rp](b, \theta) d\theta \\ &= [Rp](b_0, \theta_0) = \int_{-V_0}^{V_0} p(\sqrt{v^2 + b^2}) dv, \end{aligned}$$

where $t^2 = b^2 + v^2$, $s = v + r \cdot \sin(-\theta)$, $\psi + (-\theta) = \pi/2$, $V_0^2 = a^2 - b_0^2$, $b_0 = r \cdot \cos(-\theta_0)$, $\theta_0 \in [-\pi/2, \pi/2]$, and (2), (3), and the theorem of integral mean value of functions have been used. Both b_0 and θ_0 can be determined from the right-hand side of (6) for a given $F(r)$.

Let

$$(10) \quad w^2 = b_0^2 + v^2$$

and, transforming the variable v to w , $dv = (w/v)dw$; using (2), we have

$$\text{L.H.S.} = 2\pi \int_{b_0}^a w(w^2 - b_0^2)^{-1/2} p(w) dw.$$

In order to obtain a standard form of integral equation, let us change the variable again. Let

$$(11a) \quad u = a^2 - w^2,$$

$$(11b) \quad x = a^2 - b_0^2.$$

Changing the variable w to u , $du = -dw^2$; then the above L.H.S. becomes

$$(12) \quad \pi g(x) = \pi \int_0^x (x-u)^{-1/2} p(u) du \quad (0 \leq x \leq a^2).$$

The left-hand side of (12) should be a function of x and is denoted by $g(x)$. From (12), we should have

$$(13) \quad g(0) = 0.$$

Substituting (12) into the left-hand side of (6), we have

$$(14) \quad g(x) = (1/\pi)F(r) \quad (0 \leq x \leq a^2).$$

Now, $g(x)$ is a known function by (14), and thus (12) is a standard Abel integral equation and its solution is well known (e.g., in [4]):

$$(15) \quad p(x) = (2\pi) \left[\sqrt{x} g'(0) + \int_0^x \sqrt{x-u} g''(u) du \right] \quad (0 \leq x \leq a^2),$$

where $g' = dg/dx$, $x = a^2 - b_0^2 = a^2 - r^2 \cos^2 \theta_0$, and $\cos \theta_0$ is determined by $g(0) = 0$.

Example. The Hertz contact problem of two elastic balls is considered as an example; i.e., we take

$$(16) \quad F(r) = (\alpha - \beta r^2)/(k_1 + k_2),$$

where α , β , k_1 , and k_2 are constants, and the geometric condition of the problem gives $\alpha = 2\beta a^2$, where a is the radius of the contact base.

By (14), we have $g(x) = Ax$, $g'(x) = A$, $g'' = 0$, where

$$(17) \quad A = 2\beta/[\pi(k_1 + k_2)],$$

$$(18) \quad x = a^2 - b_0^2 = a^2 - r^2/2 \quad (0 \leq x \leq a^2).$$

Then by (15), we have

$$(19a) \quad p(x) = (2A/\pi)\sqrt{x} \quad (0 \leq x \leq a^2),$$

$$(19b) \quad p(b_0) = (2A/\pi)\sqrt{a^2 - b_0^2} \quad (0 \leq b_0 \leq a).$$

Replacing the character b_0 by r , we have

$$(20) \quad p(r) = \{4\beta/[\pi^2(k_1 + k_2)]\}\sqrt{a^2 - r^2} \quad (0 \leq r \leq a),$$

which is exactly the same as Hertz's solution [1]. (Note that the expression

$$(21) \quad p(r) = (2A/\pi)\sqrt{a^2 - r^2/2} \quad (0 \leq r \leq \sqrt{2}a),$$

obtained by substituting $x = a^2 - r^2/2$ into (19a), does not represent the radical distribution of contact pressure $p(r)$, since the domain of $p(r)$ is $(0 \leq r \leq a)$ and $p(r)$ should satisfy the condition of zero pressure at the contact boundary, i.e., $p(r = a) = 0$. Therefore (20) represents the radical distribution of contact pressure.)

3. SOLUTION OF SMALL PARAMETERED INTEGRAL EQUATION (5)

Suppose that

$$(22) \quad p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots.$$

Substituting (22) into (5) and comparing the order of ε^n , we have

$$(23) \quad \iint p_0 ds d\psi = G(r),$$

$$(24) \quad \iint p_n ds d\psi = -r \iint p_{n-1} ds \cos \psi d\psi \quad (n \geq 1),$$

where the limits of integration are the same as (1) and are omitted and p_n ($n \geq 0$) is subjected to constraints (2) and (3).

By (15), the solution p_0 of (23) is

$$(25) \quad p_0(x_0) = (2/\pi) \left[g'_0(0)\sqrt{x_0} + \int_0^{x_0} \sqrt{x_0 - u} g''_0 \cdot (u) du \right] \quad (0 \leq x_0 \leq a^2),$$

where $g_0(x_0) = (1/\pi)G(r)$, $x_0 = a^2 - r^2 \cos^2 \theta_0$, and $\cos^2 \theta_0$ is determined by $g_0(0) = 0$.

Once p_{n-1} ($n \geq 1$) has been found, then, under certain conditions (the following Existence Theorem), the right-hand side of (24) is integrable and can

be written as a known function of r so that (24) can be solved in the same way as (23). Repeating the process of solving (24), we can obtain p_n for $n \geq 1$.

In the following, we state a theorem, which guarantees the existence of p_n for $n \geq 1$.

Existence Theorem. *Suppose that the given function $g_0(x_0) = (1/\pi)G(r)$ is continuous and has continuous derivatives g'_0, g''_0 in $[0, a^2]$. Then*

(1) (23) has a continuous solution $p_0(x_0)$ in $[0, a^2]$.

(2) (24) has a continuous solution $p_n(x_n)$ in $[0, a^2]$ for $n \geq 1$, where $x_n = a^2 - r^2 \cos^2 \theta_n$, $\cos^2 \theta_n$ is determined by $g_n(0) = 0$, $g_n(x_n) = (1/\pi)F_n(r)$, and $F_n(r) = -r \iint p_{n-1} ds \cos \psi d\psi$.

Using the first integral mean value theorem of functions, changing variables, etc., we can prove this theorem. However, to conserve space, the proof is omitted.

4. EXAMPLE

Hertz's contact problem of two elastic balls is again considered as an example for the small parametered integral equation (5).

From (20), instead of r by t , we have

$$p_0(t) = (2A/\pi)(a^2 - t^2)^{1/2} \quad (0 \leq t \leq a).$$

Substituting p_0 into (24) and using (14), we have:

$$\begin{aligned} g_1(x_1) &= \frac{1}{\pi} \left[-r \int_0^\pi \int_{-\infty}^\infty p_0(t) ds \cos \psi d\psi \right] \\ (26) \quad &= -\frac{4A}{\pi^2} \int_0^\pi \int_0^V (a^2 - r^2 \sin^2 \psi - v^2)^{1/2} dv \cos \psi d\psi \\ &= K \cdot x_1 (a^2 - x_1)^{1/2}; \end{aligned}$$

$$\begin{aligned} (27) \quad g'_1(x_1) &= K(a^2 - x_1)^{1/2} - (x_1/2)(a^2 - x_1)^{-1/2}, \\ g'_1(0) &= Ka; \end{aligned}$$

$$(28) \quad g''_1(x_1) = K(a^2 - x_1)^{-1/2} + (x_1/4)(a^2 - x_1)^{-3/2},$$

where

$$(29) \quad K = -(4\sqrt{3}A/\pi^2)(1 + \pi/2),$$

$$(30) \quad x_1 = a^2 - r^2/3,$$

$t^2 = v^2 + r^2 \sin^2 \psi$, $v = s + r \cdot \cos \psi$ (Figure 1), and $x_1 = a^2 - r^2 \cos^2 \theta_1$ ($\cos^2 \theta_1 = 1/3$ is determined by $g_1(0) = 0$) have been used.

Substituting (26)–(28) into (15), we have

$$\begin{aligned} (31) \quad p_1(x_1) &= \frac{2K}{\pi} \left\{ (a^2 - x) \log \frac{\sqrt{a^2 - x_1}}{a - \sqrt{x_1}} \right. \\ &\quad \left. + \frac{3}{4} \left[\left(\frac{5x_1}{6} - \frac{a^2}{2} \right) \cdot \log \frac{-\sqrt{x_1} - a}{\sqrt{x_1} + a} + a\sqrt{x_1} \right] \right\} \end{aligned}$$

($0 \leq x_1 \leq a^2$), and so on.

Finally, the solution of (5) can be obtained by (22).

Let us make a comparison on maximum pressure $p(0)$ at the centre $r = 0$, i.e., $x = a^2$, of the contact range, between Hertz's solution $p_0(0)$ and the asymptotic solution $p(0) = p_0(0) + \varepsilon p_1(0)$ of this paper:

$$p_0(0) = (2A/\pi)a, \quad p_1(0) = \pm(3K/(2\pi))a^2.$$

If the relative error e is defined by

$$(32) \quad e = |[p(0) - p_0(0)]/p_0(0)| = |\varepsilon p_1(0)/p_0(0)|$$

and the expression of small parameter ε is listed by [2]

$$(33) \quad \varepsilon = \frac{1}{2} \left[\frac{(1 - 2\nu_1)(1 + \nu_1)E_2}{R_1} + \frac{(1 - 2\nu_2)(1 + \nu_2)E_1}{R_2} \right] \cdot \left\{ [(E_1(1 - \nu_1^2))^{-1} + (E_2(1 - \nu_2^2))^{-1}] \right\},$$

where E_1 , E_2 , ν_1 , ν_2 , and R_1 , R_2 are the elastic modulus, Poisson's ratio, and the radius of the two elastic balls respectively, then substituting $p_0(0)$, $p_1(0)$, and ε into (32), for $E_1 = E_2$, $\nu_1 = \nu_2 = 0.3$, $R_1 = R_2 = 10a$, we have $e = 7.7342\%$. This result shows that from the criterion of the more exact integral equation (5), the classical Hertz solution is not exact enough for engineering applications and the asymptotic solution is better.

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