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ON AN IDENTITY RELATED TO MULTIVALENT FUNCTIONS

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Dedicated to Professor A. W. Goodman on his 80th birthday

ABSTRACT. We prove an algebraic identity by induction. This identity is very important in the coefficient problem for analytic functions that are p-valent in the unit disk.

1. INTRODUCTION

In 1948 Goodman [1] proposed the conjecture that if

$$f(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1}$$

is regular and p-valent in $E\colon |z| < 1$, then the coefficients satisfy the inequality

$$|b_n| \le \sum_{k=1}^{p} D(p, k, n)|b_k|, \qquad 1 \le k \le p < n, \tag{2}$$

where by definition

$$D(p, k, n) = \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)}. \tag{3}$$

It was proved in [1] that if the conjecture were true it would be sharp for every selection of the p variables $|b_1|, |b_2|, \dots, |b_p|$. In a very nice paper Goodman and Robertson [2] proved that the conjecture (2) is true for the class $T(p)$, a very large class of multivalent functions. The class $T(p)$ is the natural extension of the Rogosinski class of typically-real functions to the class of typically-real functions of order p. Further, $T(p)$ includes the class of all p-valent starlike functions with real coefficients.

The proof in [2] depends on an algebraic identity symbolized by

$$D^*(p, k, n) = D(p, k, n), \qquad 1 \le k \le p < n, \tag{4}$$

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where $D^*(p, k, n)$ will be defined later. The proof of (4) given in [2] is quite sophisticated and subtle, but can also be regarded as artificial and unsatisfactory because it is not direct. In this paper we give a direct proof of (4).

2. The algebraic identity

In [2], equation (3.17) in that paper, the quantities $D^*(p, k, n)$ are defined by equating the corresponding coefficients of $|c_k^{(p)}|$ on both sides of the equation

(3.17)
$$\sum_{k=1}^{p} D^*(p, k, n)|c_k^{(p)}| = (n-p)|c_{p-1}^{(p)}| + (n-p+1)|c_p^{(p)}| + \sum_{j=p}^{n-1}(n-j)\sum_{s=1}^{p-1} D(p-1, s, j)[|c_{s+1}^{(p)}| + 2|c_s^{(p)}| + |c_{s-1}^{(p)}|],$$

where we set $c_0^{(p)} = 0$.

In this equation the absolute value signs are not needed (for our purposes) and the superscripts in $c_k^{(p)}$ are unnecessary. Henceforth we drop both of these items.

Theorem. *Equation* (4) *holds for every set of positive integers satisfying* $1 \le k \le p < n$.

3. The proof

We first regroup the terms in (3.17) by changing the range of the indices. The standard technique will give

$$\begin{aligned}\sum_{k=1}^{p} D^*(p, k, n)c_k &= (n-p)c_{p-1} + (n-p+1)c_p \\ &+ \sum_{j=p}^{n-1}(n-j)\sum_{s=2}^{p} D(p-1, s-1, j)c_s \\ &+ \sum_{j=p}^{n-1}(n-j)\sum_{s=1}^{p-1} D(p-1, s, j)2c_s \\ &+ \sum_{j=p}^{n-1}(n-j)\sum_{s=0}^{p-2} D(p-1, s+1, j)c_s.\end{aligned} \tag{5}$$

Equating coefficients in (5), will give p equations but special attention must be given when $k = p$ or $k = p-1$. Further, when $p \ge 4$, one must consider three different cases. To avoid this difficulty we extend the definition of $D(p, k, n)$ in a natural way. Let

$$D(p, 0, j) = 0 \qquad \text{all } p, j, \tag{6}$$

$$D(p-1, p, j) = \begin{cases} 1, & \text{if } j = p, \\ 0, & \text{if } j > p, \end{cases} \tag{7}$$

$$D(p-1,p+1,j)=\begin{cases}0, & \text{if } j=p,\\ -1, & \text{if } j=p+1,\\ 0, & \text{if } j>p+1.\end{cases} \tag{8}$$

Then (5), (6), (7) and (8) give

$$\begin{aligned} D^*(p,k,n)=\sum_{j=p}^{n-1}(n-j)[&D(p-1,k-1,j)\\ &+2D(p-1,k,j)+D(p-1,k+1,j)]\end{aligned} \tag{9}$$

for $1\le k\le p<n$.

We use induction on n to prove that (4) holds for all $n=p+1,p+2,\ldots$. For $n=p+1$, equation (9) gives

$$\begin{aligned} D^*(p,k,p+1)&=D(p-1,k-1,p)+2D(p-1,k,p)+D(p-1,k+1,p)\\ &=\frac{2(2p-1)!}{(p+k)!(p-k)!}\left[\frac{(k-1)(p+k)}{(p-k+1)}+2k+\frac{(k+1)(p-k)}{(p+k+1)}\right].\end{aligned} \tag{10}$$

The sum inside the brackets will give $(4kp^2+2kp)/[(p+1)^2-k^2]$, after a small computation. Hence $D^*(p,k,n)=D(p,k,n)$ when $n=p+1$.

Now assume that (4) is true for $n=p+1,p+2,\ldots,N$. Thus

$$\begin{aligned} D(p,k,n)=\sum_{j=p}^{n-1}(n-j)[&D(p-1,k-1,j)\\ &+2D(p-1,k,j)+D(p-1,k+1,j)]\end{aligned} \tag{11}$$

for $n=p+1,p+2,\ldots,N$. We must examine

$$\begin{aligned} D^*(p,k,N+1)=\sum_{j=p}^{N}(N+1-j)[&D(p-1,k-1,j)\\ &+2D(p-1,k,j)+D(p-1,k+1,j)].\end{aligned}$$

We break the first factor into two parts, 1 and $N-j$, and observe that in the second sum we have 0 when $j=N$. Consequently

$$\begin{aligned} &D^*(p,k,N+1)\\ &\quad=\sum_{j=p}^{N}[D(p-1,k-1,j)+2D(p-1,k,j)+D(p-1,k+1,j)]\\ &\quad+\sum_{j=p}^{N-1}(N-j)[D(p-1,k-1,j)\\ &\qquad\qquad+2D(p-1,k,j)+D(p-1,k+1,j)].\end{aligned}$$

By the induction hypothesis (11), the second sum is $D(p,k,N)$, so

$$\begin{aligned} &D^*(p,k,N+1)\\ &\quad=D(p,k,N)+\sum_{j=p}^{N}[D(p-1,k-1,j)+2D(p-1,k,j)\\ &\qquad\qquad+D(p-1,k+1,j)].\end{aligned} \tag{12}$$

To complete the proof, we do a second induction. In (12) we replace N by $n-1$ and define $D^{**}(p, k, n)$ by

$$D^{**}(p, k, n) = D(p, k, n-1) + \sum_{j=p}^{n-1}[D(p-1, k-1, j) + 2D(p-1, k, j) + D(p-1, k+1, j)]. \tag{13}$$

We will use induction to prove that $D^{**}(p, k, n) = D(p, k, n)$ for $n \geq p+2$ and hence conclude that $D^{*}(p, k, n) = D(p, k, n)$. We observe that with the natural extension $D(p, p, p) = 1$. Further if $k = p$ and $n = p+1$ then (10) and (13) give $D^{**}(p, p, p+1) = 1 + D(p, p, p+1) \neq D(p, p, p+1)$.

When $n = p+2$, equation (13) gives

$$\begin{aligned} D^{**}(p, k, p+2) &= D(p, k, p+1) \\ &\quad + D(p-1, k-1, p) + 2D(p-1, k, p) + D(p-1, k+1, p) \\ &\quad + D(p-1, k-1, p+1) + 2D(p-1, k, p+1) \\ &\quad + D(p-1, k+1, p+1). \end{aligned} \tag{14}$$

From (10) the second line in (14) gives $D(p, k, p+1)$. Hence

$$\begin{aligned} D^{**}(p, k, p+2) = 2D(p, k, p+1) &+ D(p-1, k-1, p+1) \\ &+ 2D(p-1, k, p+1) + D(p-1, k+1, p+1). \end{aligned} \tag{15}$$

Using the definition of $D(p, k, n)$ and dropping $0! = 1! = 1$, we have

$$\begin{aligned} D^{**}(p, k, p+2) = 2&\frac{2k(2p+1)!}{(p+k)!(p-k)![(p+1)^2-k^2]} \\ &+ \frac{2(k-1)(2p)!}{(p+k-2)!(p-k)![(p+1)^2-(k-1)^2]} \\ &+ 2\frac{2k(2p)!(p-k)}{(p+k-1)!(p-k)![(p+1)^2-k^2]} \\ &+ \frac{2(k+1)(2p)!(p-k)(p-k-1)}{(p+k)!(p-k)![(p+1)^2-(k+1)^2]}, \end{aligned}$$

or

$$D^{**}(p, k, p+2) = \frac{2(2p)!}{(p+k)!(p-k)!}[s_1 + s_2 + s_3 + s_4]. \tag{16}$$

Here by definition,

$$s_1 + s_3 = \frac{2k(2p+1)}{(p+1)^2-k^2} + \frac{2k(p^2-k^2)}{(p+1)^2-k^2} = 2k, \tag{17}$$

and

$$\begin{aligned} s_2 + s_4 &= \frac{(k-1)(p+k-1)}{p-k+2} + \frac{(k+1)(p-k-1)}{p+k+2} \\ &= \frac{2k(p^2-p+k^2-3)}{(p+2)^2-k^2}. \end{aligned} \tag{18}$$

Thus $s_1+s_2+s_3+s_4=k(2p+1)(2p+2)/[(p+2)^2-k^2]$, and hence from (16)

$$D^{**}(p,k,p+2)=\frac{2k(p+2)!}{(p+k)!(p-k)![(p+2)^2-k^2]}=D(p,k,p+2). \tag{19}$$

We now apply mathematical induction to the statement that $D^{**}(p,k,n)=D(p,k,n)$ assuming that it is true for indices $n=p+2,p+3,\dots,L$. From (13) we have

$$\begin{aligned}&D^{**}(p,k,L+1)\\ &\quad=D(p,k,L)+\sum_{j=p}^{L}[D(p-1,k-1,j)\\ &\qquad\qquad+2D(p-1,k,j)+D(p-1,k+1,j)]\end{aligned} \tag{20}$$

or

$$\begin{aligned}D^{**}(p,k,L+1)&=D(p,k,L)-D(p,k,L-1)+D(p,k,L-1)\\ &\quad+\sum_{j=p}^{L}[D(p-1,k-1,j)+2D(p-1,k,j)\\ &\qquad+D(p-1,k+1,j)].\end{aligned} \tag{21}$$

But by the induction hypothesis

$$\begin{aligned}&D(p,k,L-1)+\sum_{j=p}^{L-1}[D(p-1,k-1,j)+2D(p-1,k,j)\\ &\qquad+D(p-1,k+1,j)]\\ &\quad=D(p,k,L).\end{aligned}$$

So (21) becomes

$$\begin{aligned}D^{**}(p,k,L+1)&=2D(p,k,L)-D(p,k,L-1)+D(p-1,k-1,L)\\ &\quad+2D(p-1,k,L)+D(p-1,k+1,L)\\ &=2\frac{2k(L+p)!}{(p+k)!(p-k)!(L-p-1)!(L^2-k^2)}\\ &\quad-\frac{2k(L+p-1)!}{(p+k)!(p-k)!(L-p-2)![(L-1)^2-k^2]}\\ &\quad+\frac{2(k-1)(L+p-1)!}{(p+k-2)!(p-k)!(L-p)![L^2-(k-1)^2]}\\ &\quad+2\frac{2k(L+p-1)!(p-k)}{(p+k-1)!(p-k)!(L-p)!(L^2-k^2)}\\ &\quad+\frac{2(k+1)(L+p-1)!(p-k)(p-k-1)}{(p+k)!(p-k)!(L-p)![L^2-(k+1)^2]}.\end{aligned} \tag{22}$$

The first and fourth terms in the above expression combine to give

$$(2k)\frac{2(L+p-1)!}{(p+k)!(p-k)!(L-p)!}.$$

Then $D^{**}(p,k,L+1)$ can be written as

$$D^{**}(p,k,L+1)=\frac{2(L+p-1)!}{(p+k)!(p-k)!(L-p)!}I(p,k,L), \tag{23}$$

where

$$I(p,k,L) = 2k - \frac{k(L-p)(L-p-1)}{(L-1)^2-k^2} + \frac{(k-1)(p+k)(p+k-1)}{L^2-(k-1)^2} + \frac{(k+1)(p-k)(p-k-1)}{L^2-(k+1)^2}. \tag{24}$$

To simplify $I(p,k,L)$ we use one k and the next two terms in (24) and set

$$\Phi(p,k,L) = k - \frac{k(L-p)(L-p-1)}{(L-1)^2-k^2} + \frac{(k-1)(p+k)(p+k-1)}{L^2-(k-1)^2}. \tag{25}$$

Observing that the L.C.D. in (25) is $[(L-k)^2-1](L+k-1)$, we have

$$\begin{aligned}\Phi(p,k,L) &\\ &= k + \frac{1}{L+k-1} \\ &\cdot \left(\frac{-k(L-p)(L-p-1)(L+1-k)+(k-1)(p+k)(p+k-1)(L-1-k)}{(L-k)^2-1}\right).\end{aligned}$$

Let $\mathcal{N}$ be the numerator of the last term in Φ. We write $\mathcal{N}$ as a polynomial in p and after a moderate computation we find that

$$\begin{aligned}\mathcal{N} &= p^2(-L-k+1)+p(L+k-1)(2Lk-2k^2+1) \\ &\quad + k(L+k-1)(-L^2+2kL-L+1-k^2).\end{aligned} \tag{26}$$

After an obvious cancellation, this gives

$$\begin{aligned}\Phi(p,k,L) &= k + \frac{-p^2+(2Lk-2k^2+1)p+k(-L^2+2kL-L+1-k^2)}{(L-k)^2-1} \\ &= \frac{-p^2+(2Lk-2k^2+1)p-kL}{(L-k)^2-1}.\end{aligned} \tag{27}$$

From this last expression it is clear that $\Phi(p,k,L) = \Phi(p,-k,-L)$ and using this in (25) we find that

$$\Phi(p,-k,-L) = -k + \frac{k(L+p)(L+p+1)}{(L+1)^2-k^2} - \frac{(k+1)(p-k)(p-k-1)}{L^2-(k+1)^2}. \tag{28}$$

From (24) and (25) $I(p,k,L)$ can be put in the form

$$I(p,k,L) = k + \Phi(p,k,L) + \frac{(k+1)(p-k)(p-k-1)}{L^2-(k+1)^2}. \tag{29}$$

When $\Phi(p,k,L)$ is replaced by $\Phi(p,-k,-L)$ given by (28) we find that

$$I(p,k,L) = \frac{k(L+p)(L+p+1)}{(L+1)^2-k^2}.$$

Combining this with (23) we have

$$\begin{aligned}D^{**}(p,k,L+1) &= \frac{2(L+p-1)!}{(p+k)!(p-k)!(L-p)!} \cdot \frac{k(L+p)(L+p+1)}{(L+1)^2-k^2} \\ &= D(p,k,L+1).\end{aligned}$$

References

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