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ON UNIONS AND INTERSECTIONS OF SETS OF SYNTHESIS

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ABSTRACT. Local techniques introduced by Saeki and Stegeman are employed to give conditions for unions and finite intersections of S-sets to be S-sets.

Using local techniques we study unions and intersections of sets of spectral synthesis (S-sets). Beyond the well-known fact that the intersection of two S-sets need not be an S-set, there appears to be no known result about intersections. It is not known whether the union of two S-sets is an S-set, but there are some partial results (e.g., Saeki [2] and Warner [6]). We give simple sufficient conditions for finite intersections and (possibly infinite) unions of S-sets to be S-sets.

We generally follow the notation and terminology of Stegeman [4]. $A(G)$ denotes the Fourier algebra of a locally compact abelian group G. Suppose $x \in G$, $f, g \in A(G)$, and I, J are ideals in $A(G)$. We write

(i) $f =_x g$ if f and g agree in some neighbourhood of x;
(ii) $f \in_x I$ if $f =_x g$ for some $g \in I$;
(iii) $I \subset_x J$ if $f \in_x J$ for every $f \in I$;
(iv) $I =_x J$ if $I \subset_x J$ and $J \subset_x I$.

For a closed subset E of G, $I(E)$ and $J(E)$ denote the largest and smallest closed ideals, respectively, with hull E. $\Delta(E)$ ("the difference spectrum of E" [3]) is the set of points of nonsynthesis of E,

$$\Delta(E) = \{x \in G: I(E) \neq_x J(E)\},$$

so that E is an S-set if and only if $\Delta(E)$ is empty. $\Delta(E)$ is a subset of ∂E, the boundary of E. It was used by Saeki [2] and was later systematically exploited in [4], with different notations. We also make frequent use of the following result proved by Stegeman [4].

Theorem A (Stegeman). *If E is a closed subset of G such that $\Delta(E) \subset C \subset E$ for some C-set C, then E is an S-set.*

The basic idea of our approach is to get suitable relations between the difference spectra of the sets involved and their unions and intersections.

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1. **Lemma.** *Let* E_1, E_2 *be closed subsets of* G. *Then*

(i) $\Delta(E_1 \cap E_2) \subset \Delta(E_1) \cup \Delta(E_2) \cup (\partial E_1 \cap \partial E_2)$,
(ii) $\Delta(E_1) \cup \Delta(E_2) \subset \Delta(E_1 \cup E_2) \cup (E_1 \cap E_2)$.

Proof. Let $x \in \Delta(E_1 \cap E_2) \subset \partial(E_1 \cap E_2)$. Suppose $x \notin \partial E_1 \cap \partial E_2$. Since $\partial(E_1 \cap E_2) \subset \partial E_1 \cup \partial E_2$, as is easily checked, we have $x \in \partial E_1$ or $x \in \partial E_2$, but not both. We prove that $x \in \Delta(E_i)$ if $x \in \partial E_i$, $i = 1, 2$.

Suppose $x \in \partial E_1$. Now $x \notin \partial E_2$ and $x \in E_2$ imply that there is a neighbourhood V of x with $V \subset E_2$. Choose $k \in A(G)$ with $k = 1$ in a neighbourhood of x and $\operatorname{supp} k \subset V$. Since $x \in \Delta(E_1 \cap E_2)$, there is an $f \in I(E_1 \cap E_2)$ such that $f \notin_x J(E_1 \cap E_2)$. Then $fk \in I(E_1)$. If $x \notin \Delta(E_1)$, then $fk \in_x J(E_1)$, so $f =_x fk \in J(E_1) \subset J(E_1 \cap E_2)$, a contradiction.

Thus $x \in \partial E_1$ implies $x \in \Delta(E_1)$. Similarly, if $x \in \partial E_2$, then $x \in \Delta(E_2)$ and (i) is proved.

The proof of (ii) is similar but simpler: one proves that if $x \in \Delta(E_1)$ and $x \notin E_2$, then $x \in \Delta(E_1 \cup E_2)$.

Our first application of Lemma 1 is the following result on intersections. It shows, for instance, that an annulus is an S-set.

2. **Theorem.** *Let* E_1, E_2 *be S-sets. If there is a C-set* C *such that* $\partial E_1 \cap \partial E_2 \subset C \subset E_1 \cap E_2$, *then* $E_1 \cap E_2$ *is an S-set. In particular, the conclusion holds if* $\partial E_1 \cap \partial E_2 = \varnothing$.

Proof. By (i) of Lemma 1

$$\Delta(E_1 \cap E_2) \subset \partial E_1 \cap \partial E_2 \subset C \subset E_1 \cap E_2,$$

so the assertion is a consequence of Theorem A.

3. *Remark.* If E_1, E_2, E_3 are S-sets with $\partial E_1 \cap \partial E_2 \cap \partial E_3 = \varnothing$, it is not necessary that $E_1 \cap E_2 \cap E_3$ is an S-set. Take, for example, $E_1 = G = \mathbb{R}^3$, $E_2 =$ the closed unit ball, and $E_3 =$ the complement of the open unit ball. For a finite collection $\{E_i\}$ of S-sets, the correct version is that if $\partial E_i \cap \partial E_j = \varnothing$ for $i \neq j$, then $\bigcap E_i$ is an S-set. This again is no longer true for infinite collections: take $E_1 =$ the closed unit ball in $\mathbb{R}^3$ and $E_n = \{x \in \mathbb{R}^3 \colon \|x\| \geq 1 - \frac{1}{n}\}$, $n > 1$.

We now come to results on unions. We first mention the following result of Saeki [2].

4. **Theorem** (Saeki). *If* E_1, E_2 *are S-sets and if there is a C-set* C *such that*

$$\partial(E_1) \cap \partial(E_2) \cap \partial(E_1 \cup E_2) \subset C \subset E_1 \cup E_2,$$

then $E_1 \cup E_2$ *is an S-set.*

5. *Remark.* Theorem 4 is also stated as one part of Theorem 4′ in Warner [6], the other part being a converse, namely, if $E_1 \cup E_2$ is an S-set and if there is a C-set C satisfying the condition mentioned in Theorem 4, then E_1 and E_2 are S-sets. However, it is easy to see that this converse statement is false. Just take $E_1 = S^2$ and $E_2 =$ any closed convex set (e.g., a ball) in $\mathbb{R}^3$ containing E_1 in its interior; then $\partial E_1 \cap \partial E_2 = \varnothing$, $E_1 \cup E_2 = E_2$ is an S-set, but E_1 is not an S-set. The same example shows also that Warner's Lemma 3′ [6] is not true. We, however, have the following 'if and only if' result.

6. **Theorem.** *Let* E_1, E_2 *be closed subsets of* G. *Suppose there are* C*-sets* C_1, C_2 *such that*

$$\partial E_1 \cap E_2 \subset C_1 \subset E_1, \qquad E_1 \cap \partial E_2 \subset C_2 \subset E_2.$$

Then $E_1 \cup E_2$ *is an* S*-set if and only if* E_1 *and* E_2 *are* S*-sets.*

Proof. Since $\Delta(E_i) \subset \partial E_i$, it follows from Lemma 1(ii) that $\Delta(E_1) \subset \Delta(E_1 \cup E_2) \cup (\partial E_1 \cap E_2)$ and similarly for E_2. Theorem A now gives one half of the result, while the other half is a consequence of Theorem 4 (recalling that the union of two C-sets is a C-set).

It is well known that a closed countable union of C-sets is a C-set. Here is a result for infinite unions of S-sets.

7. **Lemma.** *Let* $\{E_i\}$ *be a collection of mutually disjoint closed sets in* G. *Suppose that, for each* j, $\bigcup_{i\neq j} E_i$ *is closed. Then*

(i) $\Delta(\bigcup E_i) \subset \bigcup \Delta(E_i)$,
(ii) $\Delta(E_j) \subset \Delta(\bigcup E_i)$ *for each* j.

Proof. Let $x \in \Delta(\bigcup E_i)$. Then $x \in E_j$ for a unique j. We prove $x \in \Delta(E_j)$. Choose a neighbourhood V of x such that $\overline{V} \cap (\bigcup_{i\neq j} E_i) = \varnothing$, and then choose a $k \in A(G)$ such that $k = 1$ near x and $k = 0$ off V. Since $x \in \Delta(\bigcup E_i)$, there is an $f \in I(\bigcup E_i)$ with $f \notin_x J(\bigcup E_i)$.

Assume $x \notin \Delta(E_j)$. Then there is a $g \in J(E_j)$ with $f =_x g$. But $gk \in J(\bigcup E_i)$, so that $f =_x fk =_x gk \in J(\bigcup E_i)$, a contradiction. Hence (i) is proved. The proof of (ii) does not need any new ideas, and we omit it.

8. **Theorem.** *Let* $\{E_i\}$ *be a collection of mutually disjoint closed sets in* G *satisfying the condition that* $\bigcup_{i\neq j} E_i$ *is closed for each* j. *Then* $\bigcup E_i$ *is an* S*-set if and only if each* E_i *is an* S*-set.*

Proof. Immediate from Lemma 7.

Using the fact that a closed set which is a countable union of C-sets is a C-set, arguments similar to the above give the following result.

9. **Theorem.** *Let* $\{E_i\}$ *be a collection of closed subsets of* G *satisfying the following conditions*:

(a) $E_i \cap E_j = \varnothing$, $i \neq j$, *except for countably many* i, j.
(b) $E_i \cap E_j$ *is a* C*-set for all* i, j, $i \neq j$.
(c) $\bigcup_{i\neq j} E_i$ *is closed for each* j.
(d) $\bigcup_{i,j,i\neq j}(E_i \cap E_j)$ *is closed.*

Then $\bigcup E_i$ *is an* S*-set if and only if each* E_i *is an* S*-set.*

10. **Corollary.** *Let* $\{E_i\}$ *be a collection of* S*-sets in* G *satisfying* (a), (b) *and* (d) *of Theorem* 9. *In place of* (c), *suppose that* $C = \bigcup_j \{\overline{\bigcup_{i\neq j} E_i} \setminus \bigcup_{i\neq j} E_i\}$ *is a* C*-set. Then* $\overline{\bigcup E_i}$ *is an* S*-set.*

Proof. $F_i = E_i \cup C$ are S-sets satisfying the conditions of Theorem 9 and $\overline{\bigcup E_i} = \bigcup F_i$.

11. *Remark.* It is easy to construct examples of S-sets using our results, e.g., the Hawaiian ear ring in $\mathbb{R}^2$ [5, pp. 111–112]. Lemma 4.3.7 of [1], which says

that $(\bigcup E_n) \cup \{0\}$ is an S-set in $\mathbb{R}$, where each $E_n \subset (\frac{1}{n+1}, \frac{1}{n})$ is an S-set, can also be deduced from the results given above.

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