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EXISTENCE OF AN ANGULAR DERIVATIVE FOR A CLASS OF STRIP DOMAINS

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ABSTRACT. A strip domain R is said to have an angular derivative if for each conformal map $\phi : R \to S = \{z : |\text{Im}\, z| < 1/2\}$ the limit $\lim(\phi(w) - w)$ exists and is finite as $\text{Re}\, w \to +\infty$. Rodin and Warschawski considerd a class of strip domains for which the euclidean area of $S \setminus R'$ is finite, where R' denotes a Lipschitz approximation of R, $R' \subset R$. They showed that a sufficient condition for an angular derivative to exist is that the euclidean area of $R' \setminus S$ be finite. We prove that this condition is also necessary.

INTRODUCTION

Let R be a simply-connected region in the plane, containing the real axis and such that ∂R meets both half-planes. Thus R contains $\pm\infty$ as prime ends. Let $\phi : R \to S$ be a one-to-one conformal map of R to $S = \{z : |\text{Im}\, z| < 1/2\}$ such that $\pm\infty$ correspond. The mapping ϕ has an angular derivative c $(-\infty < c < +\infty)$ at $+\infty$ if the following two conditions are satisfied:

$$(1) \qquad \begin{cases} \text{For each } \delta > 0 \text{ there is a } U_\delta \text{ such that} \\ R_\delta \equiv \{w : \text{Re}\, w > U_\delta,\ -1/2 + \delta < \text{Im}\, w < 1/2 - \delta\} \subset R, \end{cases}$$

$$(2) \qquad \text{for each } \delta \text{ satisfying } 0 < \delta < 1/2, \quad \lim_{\substack{\text{Re}\, w \to +\infty \\ w \in R_\delta}} (\phi(w) - w) = c.$$

If one such map ϕ has an angular derivative at $+\infty$, then so do all the others and we say that R has an angular derivative. The "problem of angular derivative" is to find euclidean geometric conditions on R such that an angular derivative exists. A moment's reflection shows that the existence or otherwise of the angular derivative pertains to how closely the given strip R matches the standard strip S near $+\infty$ in a conformal sense.

Let S_μ^λ denote the euclidean strip bounded by $\{x + iy : y = \lambda\}$ above and $\{x + iy : y = -\mu\}$ below for λ and μ positive. Given a strip R of the type specified above, let $\phi_\mu^\lambda : R \to S_\mu^\lambda$ be a conformal map so that $\pm\infty$ correspond.

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It is then trivial to check that the angular derivative at $+\infty$ can exist for ϕ_μ^λ only for at most one choice of λ, μ. By taking a suitable linear transformation of both R and S_μ^λ we normalize to the situation $\lambda = \mu = 1/2$.

The following definition of Lipshitz-1 minorants of ∂R is as in [6]. Consider Lipshitz-1 functions $f : \mathbb{R} \to \mathbb{R}$. Let B^+ be the family of these f for which $\partial R \cap \{w : \operatorname{Im} w > 0, \ \operatorname{Re} w > 0\}$ lies above the graph $\{(u, f(u)+1/2) : u \in \mathbb{R}\}$. Let B^- be the family of these f for which $\partial R \cap \{w : \operatorname{Im} w < 0, \ \operatorname{Re} w > 0\}$ lies below the graph $\{(u, f(u) - 1/2) : u \in \mathbb{R}\}$. Define $h_\pm : \mathbb{R} \to \mathbb{R}$ by

$$h_+(u) = \sup_{f \in B^+} f(u), \qquad h_-(u) = \inf_{f \in B^-} f(u).$$

Then h_+ and h_- are the Lipshitz-1 approximations to ∂R, from inside R. We prove:

Theorem 1. *Let R be as above, and assume that*

$$(3) \qquad \int_0^\infty \min(h_+(u), 0)\, du > -\infty \quad \textit{and} \quad \int_0^\infty \max(h_-(u), 0)\, du < \infty.$$

Then R has an angular derivative at ∞ if and only if

$$(4) \qquad \int_0^\infty \max(h_+(u), 0)\, du < \infty \quad \textit{and} \quad \int_0^\infty \min(h_-(u), 0)\, du > -\infty.$$

A sketch quickly demonstrates that conditions (3) and (4) have simple geometrical meaning. In fact condition (3) states that certain "inner areas", relative to S, are assumed to be finite, and condition (4) then asserts that for the existence of an angular derivative the corresponding "outer areas" must also be finite. Clearly this ties in with how well the standard strip S approximates the Lipshitz modification of R.

Rodin and Warschawski formulated the above statement in [6] and proved that (4) is sufficient for an angular derivative to exist for a strip region R which satisfies (3). We prove that condition (4) is also necessary when (3) holds.

Remark. A related result is that of Burdzy [2, Theorem 7.1] which is stated in the half-plane setting and proved using probabilistic methods. Later Carroll [3] and then Gardiner [4] gave complex analysis proofs of Burdzy's Theorem. Rodin and Warschawski [6] claim that Theorem 1 is equivalent to Burdzy's Theorem. However the author fails to see a rigorous proof of that equivalence. One difficulty is that it is not clear that Lipschitz minorants in the half-plane and strip regions correspond.

The situation when condition (3) fails is still open.

Proof. As indicated above, we only show the necessity of (4). Suppose R has an angular derivative at $+\infty$. Let θ_u denote that cross-cut of R which intersects the real axis and which lies on the vertical line having real part u. For $u_1 < u_2$, let $\lambda_R(u_1, u_2)$ be the extremal length of all arcs in R which join the cross-cuts θ_{u_1} to θ_{u_2} and lie in the component of $R - \theta_{u_1} - \theta_{u_2}$ which contains each θ_u, $u_1 < u < u_2$. Since R has an angular derivative, it follows from [7, Theorem 6] that

$$(5) \qquad \lambda_R(u_1, u_2) = u_2 - u_1 + o(1),$$

where $o(1) \to 0$ as $u_2 > u_1 \to +\infty$.

Let R' be the right half-plane portion of the strip region bounded by (and not including)

$$\beta_+ = \{(u, h_+(u) + 1/2)\}_{u\in\mathbb{R}}, \qquad \beta_- = \{(u, h_-(u) - 1/2)\}_{u\in\mathbb{R}}.$$

Note that $R' \subset R$, and by the comparison principle [1, Theorem 4.1], we have $\lambda_R(u_1, u_2) \le \lambda_{R'}(u_1, u_2)$. So that by (5),

$$\lambda_{R'}(u_1, u_2) \ge u_2 - u_1 + o(1). \tag{6}$$

We now obtain an upper bound for $\lambda_{R'}(u_1, u_2)$. To do this, consider the conjugate extremal distance $1/\lambda_{R'}(u_1, u_2)$ [1, p. 53]. We will introduce a specific metric ρ to obtain a lower bound for $1/\lambda_{R'}(u_1, u_2)$.

Lemma 1. *If condition* (3) *holds and* R *has an angular derivative at* ∞, *then* $h_\pm(u) \to 0$ *as* $u \to +\infty$.

Proof. That $\min(h_+(u), 0) \to 0$ and $\max(h_-(u), 0) \to 0$ can be seen easily from condition (1) in the definition of angular derivative and since h_+ and h_- are Lipshitz minorants to ∂R.

Further, $\max(h_+(u), 0) \to 0$ and $\min(h_-(u), 0) \to 0$ follow directly from [5, p. 102, Proposition 1], since $R' \subset R$. However for the sake of completeness we include a proof.

Suppose, by contradiction, that $\max(h_+(u), 0) \nrightarrow 0$. Hence there exist $\epsilon_0 > 0$ and $\{u_i\}$, $u_i \to +\infty$ such that $h_+(u_i) > \epsilon_0$ for all i. Set $s_i = u_i - \epsilon_0/2$, $t_i = u_i + \epsilon_0/2$. Now $\max(h_-(u), 0) \to 0$ implies that there exists U such that $\max(h_-(u), 0) < \epsilon_0/4$ for all $u \ge U$. Since h_+ is Lipshitz-1, the comparison principle [1, Theorem 4.1] shows that for all i such that $u_i \ge U + \epsilon_0$, we have $\lambda_{R'}(s_i, t_i) \le (t_i - s_i)/(1 + \epsilon_0/2 - \epsilon_0/4)$ and (6) yields that $(t_i - s_i)/(1 + \epsilon_0/4) \ge t_i - s_i + o(1)$, as $i \to \infty$. But, $t_i - s_i \equiv \epsilon_0$ and the above is a contradiction. Hence $h_+(u) \to 0$ as $u \to +\infty$. Similarly we can show that $h_-(u) \to 0$.

With no loss of generality we assume $|h_\pm(u)| \le 1/8$ for $u \ge 0$. To enable us to define ρ, we set for $u \in \mathbb{R}$

$$h_+^*(u) = \begin{cases} 2h_+(u), & \text{if } h_+(u) < 0, \\ -\frac{1}{2}h_+(u), & \text{if } h_+(u) \ge 0 \end{cases}$$

and

$$h_-^*(u) = \begin{cases} 2h_-(u), & \text{if } h_-(u) > 0, \\ -\frac{1}{2}h_-(u), & \text{if } h_-(u) \le 0. \end{cases}$$

Let R^* be the right half-plane portion of the strip region bounded by (and not including)

$$\beta_+^* = \{(u, h_+^*(u) + \tfrac{1}{2})\}_{u\in\mathbb{R}}, \qquad \beta_-^* = \{(u, h_-^*(u) - \tfrac{1}{2})\}_{u\in\mathbb{R}}.$$

Note that $R^* \subset R' \cap S$. Let $0 < u_1 < u_2$ and $Q = Q(u_1, u_2) = R' \cap \{u + iv : u_1 < u < u_2\}$,

$$\begin{aligned} R_1 &= \{u + iv \in Q : h_+(u) > 0, v > 0\}, \\ R_2 &= \{u + iv \in Q : h_-(u) < 0, v < 0\}, \\ R_3 &= \{u + iv \in Q : h_+(u) < 0, v > 0\}, \\ R_4 &= \{u + iv \in Q : h_-(u) > 0, v < 0\}. \end{aligned}$$

Define a metric ρ on $Q(u_1, u_2)$:

$$\rho(u+iv) = \begin{cases} 1, & u+iv \in \bar{R}^* \cap Q, \\ 1/2, & u+iv \in (R_1 \cup R_2) - \bar{R}^*, \\ 2\sqrt{2}, & u+iv \in (R_3 \cup R_4) - \bar{R}^*. \end{cases} \tag{7}$$

Clearly ρ is well defined and Borel measurable. Let $\Gamma = \Gamma(u_1, u_2)$ be the family of all connected, rectifiable arcs which have one end-point on β_+, one end-point on β_-, and all other points in Q. By [1, 4-1, 4-2], it is evident that

$$\frac{1}{\lambda_{R'}(u_1, u_2)} = \lambda_Q(\Gamma). \tag{8}$$

Lemma 2. *If ρ is as above, then*

$$L(\Gamma, \rho) \equiv \inf_{\gamma \in \Gamma} L(\gamma, \rho) = \inf_{\gamma \in \Gamma} \int_\gamma \rho\, ds \geq 1.$$

Proof. Let $\gamma \in \Gamma$. Clearly it is enough to restrict oneself to simple arcs γ (i.e., with no self-intersections). Parametrize γ as $\gamma : [0, 1] \to Q$ so that it starts on β_+. Let

$$l = \inf\{t : 0 < t < 1, \gamma(t) \in \mathbb{R}\}.$$

Denote by α the curve $\gamma|[0, l]$ with the orientation reversed, where by $\gamma|[0, l]$ we mean γ restricted to $[0, l]$. Thus $\alpha : [0, l] \to Q \cap \{\text{Im } w \geq 0\}$ is a curve which starts on the real axis and ends on β_+.

It suffices to show that $\int_\alpha \rho\, ds \geq 1/2$. Note that if the euclidean length of α is ≥ 1, we are done, for $\rho \geq 1/2$ in Q. So we assume that the euclidean length of α is < 1. Label the components of the open set $R_1 - \bar{R}^*$ as $A_1, A_2, \ldots$, and those of $R_3 - \bar{R}^*$ as $B_1, B_2, \ldots$. Let $D = (\cup_i A_i) \cup (\cup_j B_j)$, so that D is open.

If $\alpha((0, l)) \cap D = \varnothing$, then $\alpha(l) = \lim_{t \to l} \alpha(t) \in \beta_+ \cap \beta_+^* \subset \{\text{Im } w = 1/2\}$. Further $\alpha((0, l)) \subset \bar{R}^*$, so by (7) there is nothing to prove. So for the rest of the proof we assume $\alpha((0, l)) \cap D \neq \varnothing$. We define numbers t_k, s_k and set(s) C_k, $k = 1, 2, \ldots$, as follows:

Set $s_0 = t_0 = 0$, $C_0 = \varnothing$, and define

$$t_1 = \inf\{t : t > 0, \alpha((0, t)) \cap D \neq \varnothing\}.$$

We assert that there exists a unique component C_1 of D such that $\alpha(t_1) \in \partial C_1 \cap \beta_+^*$, here $C_1 = A_{i_0}$ for some i_0, or $C_1 = B_{j_0}$ for some j_0. Indeed it is not difficult to check that $\alpha(t_1) \in \partial D \cap \partial R^* \subset \beta_+^*$. Further if $\alpha(t_1)$ lies on the boundary of two components of D, then $\alpha(t_1) \in \beta_+ \cap \beta_+^*$, and so $t_1 = l$, which is a contradiction. Hence $\alpha(t_1)$ detemines C_1 uniquely.

We now define recursively, for $k \geq 1$, the following :

If $\alpha((t_k, l)) \cap (D - C_k) = \varnothing$, we set

$$s_k = \sup\{t : t_k < t < l,\ \alpha(t) \in \partial C_k \cap \beta_+^*\} \tag{9}$$

and stop. Otherwise we define

$$t_{k+1} = \inf\{t : t > t_k,\ \alpha((t_k, t)) \cap (D - C_k) \neq \varnothing\},$$

$$s_k = \sup\{t : t_k < t < t_{k+1},\ \alpha(t) \in \partial C_k \cap \beta_+^*\},$$

and C_{k+1} as the unique component of D such that $\alpha(t_{k+1}) \in \partial C_{k+1}$. Increment k by 1 and proceed with the recursion.

The proof that C_{k+1} exists uniquely is exactly similar to the corresponding proof for C_1.

Note that $\alpha(t_k), \alpha(s_k) \in \partial C_k \cap \beta_+^*$, and if $\{s_k\}$ is an infinite sequence, then $0 < t_1 < s_1 < t_2 < s_2 < \cdots$. If $\{s_k\}$ is a finite sequence, say $\{s_1, \cdots, s_n\}$, then $0 < t_1 < s_1 < \ldots < t_n \leq s_n$.

Let k be such that $\alpha(t_k)$, $\alpha(s_k)$ are defined.

We compute $\int_{\alpha|[0,s_k)} \rho\, ds$. Decompose $\alpha|[0, s_k)$ as follows:

$$\begin{aligned} \alpha_1 &= \alpha|[0, t_1), & \alpha_1' &= \alpha|[t_1, s_1), \\ &\vdots & &\vdots \\ \alpha_k &= \alpha|[s_{k-1}, t_k), & \alpha_k' &= \alpha|[t_k, s_k), \end{aligned}$$

so that

$$\int_{\alpha|[0,s_k)} \rho\, ds = \left(\int_{\alpha_1} + \int_{\alpha_1'} + \cdots + \int_{\alpha_k} + \int_{\alpha_k'}\right) \rho\, ds. \tag{10}$$

Let $\alpha(t_j) = u_j + iv_j$ and $\alpha(s_j) = \xi_j + i\eta_j$, for $j = 1, \ldots, k$. By the definition of t_j's and s_j's one sees that each $\alpha_j \subset \bar{R}^*$, so that by (7),

$$\int_{\alpha_j} \rho\, ds \geq |\alpha(t_j) - \alpha(s_{j-1})| \geq |v_j - \eta_{j-1}|. \tag{11}$$

For the α_j''s we note that if:

(i) $C_j = B_{i_0}$, for some i_0, then $\alpha_j' \cap (D - B_{i_0}) = \varnothing$. Thus on α_j', $\rho \geq 1$, and we have

$$\int_{\alpha_j'} \rho\, ds \geq |\eta_j - v_j|. \tag{12}$$

(ii) $C_j = A_{i_0}$, for some i_0. Then $\alpha(t_j), \alpha(s_j) \in \partial A_{i_0} \cap \beta_+^*$, which is a Lipshitz-(1/2) curve. This means $|\eta_j - v_j| \leq (1/2)|\xi_j - u_j|$, and since $\rho \geq 1/2$, this yields that

$$\int_{\alpha_j'} \rho\, ds \geq \frac{1}{2}((\eta_j - v_j)^2 + (\xi_j - u_j)^2)^{1/2} \geq \frac{\sqrt{5}}{2}|\eta_j - v_j| > |\eta_j - v_j|. \tag{13}$$

Now using (10), (11), (12), and (13) we get, since $\eta_0 = 0$, that

$$\begin{aligned} \int_{\alpha|[0,s_k)} \rho\, ds &\geq \sum_{j=1}^{k} |v_j - \eta_{j-1}| + \sum_{j=1}^{k} |\eta_j - v_j| \\ &\geq \sum_{j=1}^{k} (v_j - \eta_{j-1}) + \sum_{j=1}^{k} (\eta_j - v_j) \\ &= \eta_k. \end{aligned} \tag{14}$$

We now consider two cases:

Case I: The sequence $\{s_k\}$ is infinite.

We claim that in this case $\operatorname{Im}(\alpha(s_k)) = \eta_k \to 1/2$. For if not, then we can show that α is nonrectifiable. Indeed, suppose by contradiction, that there exists $\tau > 0$ such that

$$\eta_k < 1/2 - \tau, \tag{15}$$

for infinitely many values of k. For any such k consider $\alpha(s_k) = \xi_k + i\eta_k \in \partial C_k \cap \beta_+^*$ and $\alpha(s_{k+1}) = \xi_{k+1} + i\eta_{k+1} \in \partial C_{k+1} \cap \beta_+^*$, where $\xi_{k+1} \neq \xi_k$. Without loss of generality let $\xi_{k+1} < \xi_k$. Then there exists $\xi + i/2 \in \beta_+^*$, such that $\xi_{k+1} < \xi < \xi_k$. Since β_+^* is a Lipshitz-2 curve in general, and $\xi + i/2$, $\xi_k + i\eta_k \in \beta_+^*$, we have $1/2 - \eta_k < 2(\xi_k - \xi)$. Combining this and (15),

$$|\alpha(s_{k+1}) - \alpha(s_k)| > |\xi_{k+1} - \xi_k| > \xi_k - \xi > \tau/2,$$

for infinitely many values of k. This implies α is nonrectifiable, which is a contradiction.

Hence $\eta_k \to 1/2$. Now by (14), for every k, $\int_\alpha \rho\, ds \geq \int_{\alpha|[0, s_k)} \rho\, ds \geq \eta_k$, and since the right side tends to 1/2, we are done in this case.

Case II: The sequence $\{s_k\}$ is finite, say $\{s_1, \dots, s_n\}$.

We know by (14) that

$$\int_{\alpha|[0, s_n)} \rho\, ds \geq \eta_n. \tag{16}$$

Suppose first that $s_n = l$; then by the definition of s_n, (9), it is easy to see that $\eta_n = \operatorname{Im}(\alpha(l)) = 1/2$ and by (16) we are done.

We consider next the situation $s_n < l$. All we need to do in view of (16) is to show $\int_{\alpha|[s_n, l)} \rho\, ds \geq \frac{1}{2} - \eta_n$.

Now if $\alpha([s_n, l)) \subset \bar{R}^*$, there is nothing to prove. If $\alpha([s_n, l)) \subset C_n$, then let $\xi_n + i(\eta_n + \eta)$ be the point on β_+ directly above $\xi_n + i\eta_n$. Since β_+ is a Lipschitz-1 curve, the cone C with vertex at $\xi_n + i(\eta_n + \eta)$ in the $w = u + iv$-plane given by $C: \{v = -|u - \xi_n| + (\eta_n + \eta)\}$ always lies below β_+. Hence $\alpha|[s_n, l)$ must intersect this cone before terminating at β_+. The shortest euclidean distance from $\xi_n + i\eta_n$ to the cone is $\eta/\sqrt{2}$. Thus if:

(a) $\alpha([s_n, l)) \subset R_1$, then $\eta = 3|h_+(\xi_n)|/2$, so that

$$\int_{\alpha|[s_n, l)} \rho\, ds \geq \rho\eta/\sqrt{2} \geq 3|h_+(\xi_n)|/4\sqrt{2} \geq |h_+(\xi_n)|/2 = \frac{1}{2} - \eta_n;$$

(b) $\alpha([s_n, l)) \subset R_3$, then $\eta = |h_+(\xi_n)|$ and

$$\int_{\alpha|[s_n, l)} \rho\, ds \geq \rho\eta/\sqrt{2} = 2|h_+(\xi_n)| = \frac{1}{2} - \eta_n.$$

This proves the lemma.

Using [1, Definition 4-1] and (8) we have that

$$\frac{1}{\lambda_{R'}(u_1, u_2)} = \lambda_Q(\Gamma) = \sup_\rho \frac{L^2(\Gamma, \rho)}{A(Q, \rho)}$$

where $A(Q, \rho) = \int_Q \rho^2\, du\, dv$ and ρ is nonnegative, Borel measurable and subject to the condition $0 < A(Q, \rho) < \infty$.

Clearly ρ as defined in (7) satisfies these conditions and we have from Lemma 2 that

$$\lambda_{R'}(u_1, u_2) \le A(Q, \rho). \tag{17}$$

We now compute $A(Q, \rho)$ explicitly, using (7):

$$\begin{aligned}
A(Q, \rho) &= A(\bar{R}^* \cap Q, \rho) + A((R_1 \cup R_2) - \bar{R}^*, \rho) + A((R_3 \cup R_4) - \bar{R}^*, \rho) \\
&= \{u_2 - u_1 \quad - \frac{1}{2}\int_{u_1}^{u_2} \max(h_+(u), 0)\,du + 2\int_{u_1}^{u_2} \min(h_+(u), 0)\,du \\
&\qquad + \frac{1}{2}\int_{u_1}^{u_2} \min(h_-(u), 0)\,du - 2\int_{u_1}^{u_2} \max(h_-(u), 0)\,du\} \\
&\qquad + \{\frac{1}{4}\int_{u_1}^{u_2} \frac{3}{2}\max(h_+(u), 0)\,du \\
&\qquad\quad - \frac{1}{4}\int_{u_1}^{u_2} \frac{3}{2}\min(h_-(u), 0)\,du\} \\
&\qquad + \{-8\int_{u_1}^{u_2} \min(h_+(u), 0)\,du + 8\int_{u_1}^{u_2} \max(h_-(u), 0)\,du\} \\
&= u_2 - u_1 \quad - \frac{1}{8}\int_{u_1}^{u_2} \max(h_+(u), 0)\,du - 6\int_{u_1}^{u_2} \min(h_+(u), 0)\,du \\
&\qquad + \frac{1}{8}\int_{u_1}^{u_2} \min(h_-(u), 0)\,du + 6\int_{u_1}^{u_2} \max(h_-(u), 0)\,du.
\end{aligned}$$

From (6) and (17) we see that $A(Q, \rho) \ge u_2 - u_1 + o(1)$. By the hypothesis (3), both $\int_{u_1}^{u_2} \max(h_-(u), 0)\,du$ and $\int_{u_1}^{u_2} -\min(h_+(u), 0)\,du \to 0$ as $u_2 > u_1 \to +\infty$. Hence

$$0 \ge -\frac{1}{8}\int_{u_1}^{u_2} \max(h_+(u), 0)\,du + \frac{1}{8}\int_{u_1}^{u_2} \min(h_-(u), 0)\,du \ge o(1),$$

which readily implies (4) and hence proves the theorem.

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