AN EXPLICIT FAMILY OF EXOTIC CASSON HANDLES

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ABSTRACT. This paper contains a proof that the Casson handle that contains only one, positive, self-intersection on each level, CH^+, is exotic in the sense that the attaching circle of this Casson handle is not smoothly slice in its interior. The proof is an easy consequence of L. Rudolph's result (Bull. Amer. Math. Soc. (N.S.) **29** (1993), 51–59) that no iterated positive untwisted doubles of the positive trefoil knot is smoothly slice. An explicit infinite family of Casson handles is constructed by using the non-product h-cobordism from Ž. Bižaca (*A handle decomposition of an exotic* $\mathbb{R}^4$, J. Differential Geom. (to appear)), CH_n, $n \geq 0$, such that CH_0 is the above-described CH^+ and each CH_{n+1} is obtained by the reimbedding algorithm (Ž. Bižaca, *A reimbedding algorithm for Casson handles*, Trans. Amer. Math. Soc. **345** (1994), 435–510) in the first six levels of CH_n. An argument that no two of those exotic Casson handles are diffeomorphic is outlined, and it mimics the one from S. DeMichelis and M. Freedman (J. Differential Geom. **17** (1982), 357–453).

The purpose of this paper is to present explicit examples of exotic Casson handles. For a description of Casson handles the reader is referred to [C], [F], or [K2]. The notion of 'exoticness' is defined below. A way to describe any handle, and so a Casson handle in particular, is to think of it as a pair consisting of a manifold and a selected piece of its boundary, "the attaching area". One of the biggest break-throughs in four-dimensional topology was M. Freedman's proof [F] that every Casson handle is homeomorphic to the standard *open* 2-handle, $(D^2 \times \mathbb{R}^2, S^1 \times \mathbb{R}^2)$. Andrew Casson constructed these manifold pairs in an attempt to mimic the proof of the h-cobordism theorem for higher-dimensional manifolds in the setting of four-dimensional manifolds [C]. The piece of the higher-dimensional proof that does not apply to dimension four is the Whitney trick (see [RS, §6]). Casson handles were constructed in places where embedded 2-handles, or "Whitney discs", are needed in order to perform the Whitney trick [C]. Therefore, a consequence of Freedman's result is the *topological* h-cobordism theorem for four-dimensional manifolds [F]; that is, an h-cobordism between two simply-connected closed four-dimensional manifolds, say M_0 and M_1, is necessarily *homeomorphic* to the product cobordism, $M_0 \times [0, 1]$.

If all Casson handles were diffeomorphic to the standard open 2-handle, the

h-cobordism theorem would be *smoothly* true [C]; any smooth h-cobordism be-
tween simply-connected closed four-manifolds would be diffeomorphic to the
product cobordism. By now there are many known counterexamples to the
smooth h-cobordism theorem (for some concrete examples see [A] or [Ko]).
Obviously any Casson handle that appears as the only Whitney disc in a non-
product h-cobordism cannot be diffeomorphic to the standard open 2-handle.
Also, it cannot happen that the attaching circle of such a Casson handle bounds
a smoothly embedded disc inside the Casson handle. It is customary to refer to
a Casson handle with this non-embedding property as an *exotic Casson handle.*

It is still an open conjecture that *all* Casson handles are exotic. R. Gompf has
shown that there are infinitely many diffeomorphism classes of Casson handles
(see [G1] and [G2]). However, until recently it was not known for any particular
Casson handle whether it is exotic or not. The first concretely described example
of an exotic Casson handle was constructed in [B1]. The same Casson handle
was used in a construction of a handlebody of an exotic $\mathbb{R}^4$ [B2]. (An "exotic
$\mathbb{R}^4$" is a smooth manifold homeomorphic but not diffeomorphic to the standard
Euclidean four-space, $\mathbb{R}^4$.)

Recall that a Casson handle can be constructed as an infinite tower whose
levels consist of "kinky handles". Each kinky handle is a manifold pair diffeo-
morphic to a relative neighborhood of an immersed disc in the four-ball. A
kinky handle is completely determined by two numbers: the number of pos-
itive and the number of negative self-intersections (or "kinks") on the given
immersed disc, or "core". A new level of a tower is formed by attaching kinky
handles to cap loops generating the fundamental group of the previous level.
For each self-intersection there is a corresponding generator. Also, each of the
loops involved is assumed to have a certain canonical framing. The information
needed to reconstruct a Casson handle up to diffeomorphism can be encoded in
an infinite-based signed tree where each vertex corresponds to a self-intersection,
its sign corresponds to the sign of the self-intersection and all edges going up
from a given vertex determine the self-intersections of the kinky handle on the
next level that was attached over the loop for the given self-intersection. If such
a tree for a Casson handle is homeomorphically embedded as a subtree of a tree
for another Casson handle, then the Casson handle with the bigger tree can be
embedded in the Casson handle with the smaller tree. Such an embedding can
be constructed by adding new self-intersection points and new kinky handles to
the cores in the simpler Casson handle, as prescribed by the bigger tree. (We
will always assume that an embedding between handles preserves attaching ar-
eas or that it at least preserves attaching circles and their framings.) Similarly,
every Casson handle can be embedded in the standard 2-handle by introducing
self-intersections and kinky handles to the core of the standard 2-handle. Of
course, for exotic Casson handles there are no embeddings going in the opposite
direction. The example of an exotic Casson handle from [B1] corresponds to
a huge tree where the number of branchings grows superexponentialy with the
level. The following example is as simple as possible.

Theorem A. *The Casson handle with only one self-intersection on each level and
with only positive self-intersections, CH^+, is exotic.*

Remark. It is not known at present whether the condition that all self-inter-
sections are positive can be dropped from Theorem A. By taking mirror images
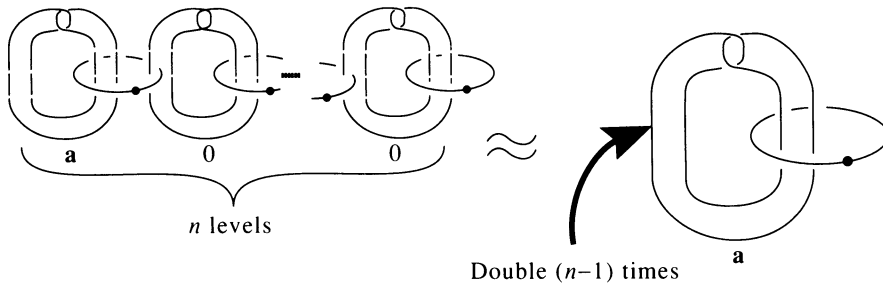
FIGURE 1. Equivalent link calculus pictures of the first n levels of CH^+.

it follows that the Casson handle CH^-, with one negative self-intersection on each level, is also exotic. If Theorem A is true for Casson handles with one self-intersection of an arbitrary sign on each level, then *all* Casson handles are exotic. Namely, a Casson handle that embeds in an exotic Casson handle is itself exotic and any Casson handle can be embedded in one whose corresponding tree has only one self-intersection on each level. Notice that by writing the numbers from the unit interval, $[0, 1]$, in the binary notation, one can obtain a surjection from the set of Casson handles with only one kink on each level onto the interval $[0, 1]$.

Proof of Theorem A. Suppose that CH^+ is not exotic, i.e., that its attaching circle is smoothly slice. CH^+ has the limit topology as an infinite union of kinky handles and therefore the slicing disc (being compact) meets only finitely many levels; say it is embedded in the first n levels of CH^+, for some positive integer n.

Two link calculus pictures (see [K1], [K2], or [F]) of the first n levels of CH^+ are visible in Figure 1. The standard one on the left ([C], [K2], or [F]) consists of the n Whitehead links, each but the first one formed from a 1-handle (dotted circles) and a 0-framed 2-handle. The component labeled with 'a' is the attaching circle of the Casson handle CH^+ for which we assumed that it bounds a smoothly embedded disc inside the handlebody. Contracting $(n - 1)$ many complementary pairs of 1- and 2-handles, all but the first and last components of the link disappear and the first component is doubled $(n - 1)$ times, that is, the result is an $(n - 1)$-fold untwisted positive double of the Whitehead link. The assumption that the attaching circle is slice in the first n levels of CH^+ implies that this link is smoothly slice. The manifold described by Figure 2 is obtained by adding a 2-handle that is complementary to the 1-handle from Figure 1, so both can be removed from the picture and the resulting manifold is just a four-ball. The framing of the attached 2-handle is -1, which results in a right-hand twist being added to the attaching circle. The result is visible as a right-hand trefoil in Figure 3, which is what the picture would be if n was equal to one. In general, since the doubling commutes with the twisting, the picture is $(n - 1)$-fold untwisted positive double of the trefoil knot. Since this circle bounded an embedded disc before we added a -1-framed 2-handle, the resulting knot bounds the same embedded disc in the obtained manifold ($=$ four-ball), or, in other words, $(n - 1)$-fold untwisted positive double of the positive trefoil knot is smoothly slice. But this is in contradiction with
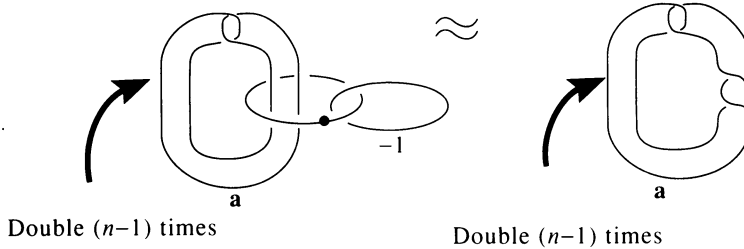
Double $(n-1)$ times   Double $(n-1)$ times

FIGURE 2   FIGURE 3

L. Rudolph's result [R] that no iterated untwisted positive double of the positive trefoil knot is smoothly slice. ☐

We will now show how to use Theorem A to provide an infinite sequence of (potentially explicitly described) exotic Casson handles, of which CH^+ is the first. In [DF], DeMichelis and Freedman analyzed Kotschick's [Ko] example of a non-product smooth h-cobordism between a pair of non-diffeomorphic one-connected four-dimensional manifolds that are distinguished by having a different value of a gauge-theoretic invariant. They have shown that the two non-diffeomorphic manifolds in the boundary of the cobordism differ by an embedding of an open manifold R, an exotic $\mathbb{R}^4$. Furthermore, in [DF] it was shown that this Kotschick's invariant can be defined for manifolds with ends periodic in the sense of [T]. Following an argument in [T], one can conclude that there is a compact K in R such that if R_n is an open smooth proper submanifold of R with the property that $K \subset R_n \subset R$ then pairs (R_n, K) and (R, K) cannot be diffeomorphic by a diffeomorphism that is identity on the compact K, and also, there cannot be an embedding $R \hookrightarrow R_n$ that is the identity on K. Roughly the argument is as follows: a compact K is chosen so that the h-cobordism is smoothly a product over the complement of K. If there was such a diffeomorphism or an embedding in the opposite direction between the manifold pairs (R_n, K) and (R, K), then there would exist a self-embedding of (R, K) that is compactly supported and equal to the identity on K. As argued in [DF], an iteration of this self-embedding and the removal of the intersection of the images would produce an open manifold with periodic end, embedded in the complement of K. This open manifold would have the same value of Kotschick's invariant as the initial, closed manifold [DF]. This construction can be done in both components of the boundary of the cobordism and since the resulting open manifolds are diffeomorphic, then the invariant has to be the same for the closed manifolds. This is not true for the given example, therefore there is no compactly supported self-embedding of (R, K).

Another explicit example of a non-product h-cobordism, due to S. Akbulut [A], was used in [B2] to produce a handlebody for an exotic $\mathbb{R}^4$. In this example a different gauge-theoretic invariant is used to distinguish between two manifolds in the boundary of the cobordism, namely a Donaldson polynomial evaluated on fixed second homology classes. However, an argument analogous to the one outlined above can be applied to relate this invariant of closed manifolds with open manifolds with periodic end, obtained by the process above.

The same non-existence conclusions apply for the similar pairs (R_n, K) and (R, K).

We continue with a rough outline of the construction of an exotic $\mathbb{R}^4$ from [B2]. The construction starts with a compact manifold with boundary, denoted by Y, whose homotopy is generated by two 1-handles. If those generators are capped by copies of a specific Casson handle the resulting manifold is an exotic $\mathbb{R}^4$ [B2, Figures 2 and 4]. The corresponding tree of the Casson handle may be explicitly determined by a sequence of nonnegative integers, but what guarantees that the construction will produce an exotic $\mathbb{R}^4$ is that the used Casson handle can be embedded in the first eight levels of CH^+. A fixed compact manifold K is constructed by capping the two generators of Y by copies of closure of the first three levels of CH^+. This guarantees that the interior of K contains an exotic $\mathbb{R}^4$ and that the cobordism from [A] is a product over the complement of K. Then, the reimbedding algorithm from [B1] produces a Casson handle embedded in the first six levels of CH^+. Call the resulting Casson handle CH_1. Two copies of CH_1 are added to $K \cup (\partial K \times [0, \infty))$ by identifying the open kinky handle (with only one, positive, self-intersection) from the third level of CH^+ in K with the first level of CH_1, which is possible since the reimbedding algorithm always preserves the first level. The resulting open smooth submanifold is denoted by R_1 and contains K. Because R_1 can be embedded in $K \cup 2$ (first eight levels of CH^+), R_1 is an exotic $\mathbb{R}^4$. The process can be repeated to construct $R_2, R_3, \ldots$ iteratively: R_{n+1} is constructed by finding a Casson handle CH_{n+1} with the property that it can be embedded with compact support in CH_n; for example, the reimbedding algorithm can be used with the first six levels of CH_n as the input. Then as before, $R_{n+1} = K \cup (\partial K \times [0, \infty)) \cup_2 CH_{n+1}$ and $K \subset R_{n+1} \subset R_n$. It is clear from the argument above that there cannot be a diffeomorphism fixed on K between different pairs (R_n, K) and (R_m, K). Also all embeddings can go only in the one direction; if $m > n$, then there is no embedding $R_m \hookrightarrow R_n$ that is the identity on K. Any diffeomorphism between CH_n and CH_m or an embedding of CH_n in CH_m, for $n < m$, can be extended over R_n and R_m by taking it to be identity over K. Therefore we have proved the following corollary.

Corollary B (Compare with [G2, Proposition 4.1]). *Let $\{CH_n\}_{n=0}^{\infty}$ be a sequence of Casson handles such that*:

   (i) $CH_0 = CH^+$;
   (ii) CH_1 *embeds in the first six levels of CH_0; and*
   (iii) CH_{n+1} *embeds in a compact subset of CH_n.*

Then, for $k < n$, CH_n does not embed in CH_k and in particular, no CH_n and CH_k are diffeomorphic for different k and n.

Remark. The existence of a Casson handle that might play the role CH_0 in Corollary B was proven by Gompf [G2] and the novelty here is an explicit description of such an CH_0. Note that by using the reimbedding algorithm from [B1] one can produce in Corollary B a sequence $\{CH_n\}_{n=0}^{\infty}$ of explicitly described Casson handles.

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