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DARBOUX'S LEMMA ONCE MORE

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ABSTRACT. Darboux's lemma states that a closed nondegenerate two-form Ω, defined on an open set in $\mathbb{R}^{2n}$ (or in a $2n$-dimensional manifold), can locally be given the form $\sum dq_i \wedge dp_i$, in suitable coordinates, traditionally denoted by $q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n$. There is an elegant proof by J. Moser and A. Weinstein. The author has presented a proof that was extracted from Carathéodory's book on Calculus of Variations. Carathéodory works with a (local) "integral" of Ω, that is, with a one-form α satisfying $d\alpha = \Omega$. It turns out that the proof becomes much more transparent if one works with Ω itself.

As in [3] we start by writing Ω (locally) as $\sum_1^N df_i \wedge dg_i$, with some functions $f_1, f_2, \dots, f_N, g_1, g_2, \dots, g_N$, and with $N \geq n$ of course. (For this step we take an integral α of Ω and write it as $\sum_1^N f_i dg_i$.) We now try to reduce N, if it is larger than n.

Since Ω^n is not 0, some n of the terms in the sum for Ω must have nonzero exterior product; the corresponding f's and g's can then be taken as coordinates $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n$ and we can write $\Omega \sum_1^n du_i \wedge v_i + \sum_{n+1}^N df_j \wedge dg_j$. To this situation we apply a standard classical and basic proposition of Hamiltonian transformation theory [5].

Let ω be a closed nondegenerate two-form on an open set in a manifold M^{2n} (with local coordinates x_i when needed), and let H be a "time-dependent Hamiltonian", i.e., a function $H(x, t)$ on $M \times \mathbb{R}$ (or on a suitable open subset thereof). Write ω_H for the two-form $\omega - dH \wedge dt$ (here ω has been pulled back to $M \times \mathbb{R}$ and t is the standard coordinate on $\mathbb{R}$).

Proposition. *There exists a (local) diffeomorphism F of $M \times \mathbb{R}$ "over $\mathbb{R}$", i.e., of the form $x' = F(x, t)$, $t' = t$ (or, briefly, of the form $x' = F(x, t)$) with inverse $x = G(x', t)$ such that*

$$F^* \omega_H = \omega \quad (\textit{and } G^* \omega = \omega_H).$$

One says that "H has been reduced to 0 by F". As a matter of fact, F is simply the expression for the solutions of the associated "canonical equations" in terms of the initial values for $t = 0$. The proof is a simple computation; we bring it, for completeness, at the end.

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We apply the proposition to Ω above by using $\sum du_i \wedge dv_i$ as ω and $-f_N$ as H. Thus we have functions $u_i' = \phi_i(u_j, v_j, t)$, $v_i' = \psi_i(u_j, v_j, t)$ such that the equation

$$\sum du_i \wedge dv_i + df_N \wedge dt = \sum d\phi_i \wedge d\psi_i$$

holds identically in (u_j, v_j, t). We now substitute the function g_N, from the expression for Ω, for t in this equation (i.e., we take the pullback of the equation under the embedding of M into $M \times \mathbb{R}$ via $x \mapsto (x, g_N(x))$. Thus $\sum du_i \wedge dv_i + df_N \wedge dg_N$ equals $\sum d\Phi_i \wedge d\Psi_i$, where $\Phi_i(u_j, v_j)$ means $\phi_i(u_j, v_j, g_N(u_j, v_j))$ and similarly for Ψ_i. So $\Omega = \sum du_i \wedge dv_i + \sum_{n+1}^{N} df_j \wedge dg_j$ equals $\sum_1^n d\Phi_i \wedge d\Psi_i + \sum_{n+1}^{N-1} df_j \wedge dg_j$, and so the number of terms in the expression for Ω has been reduced by 1. Darboux's lemma follows by iteration. □

Now we prove the proposition. We express the usual canonical differential equations of Hamiltonian theory in the language of exterior forms: A vector field $\widetilde{X}$ on $M \times \mathbb{R}$ (or on an open subset thereof) will be called Hamiltonian (to H) if

(a) it is of the form (X, ∂_t), where ∂_t is the standard vector field $\mathbb{R}$ (thus $\partial_t f = f'$) and where X at any point (x, t) is tangent to $M \times t$, so that X is a "time-dependent vector field" on M; and

(b) the substitution operator $i_{\widetilde{X}}$ nullifies the form $\omega_H = \omega - dH \wedge dt$.

(For any vector field Y the operator i_Y operates on an exterior form π by substituting Y into the first slot to π. It is characterized by three properties: (1) it nullifies functions (i.e., 0-forms); (2) one has $i_Y dh = dh(Y) = Y.h$ for any function h; (3) it is a (graded) derivation: $i_Y(\lambda \wedge \mu) = i_Y\lambda \wedge \mu + (-1)^{\deg \lambda}\lambda \wedge i_Y\mu$.)

We split the differential dH into its M- and $\mathbb{R}$-components (defined by restriction to the M- or $\mathbb{R}$-factor at (x, t)); we write this as $dH = d_M H + H_t dt$. The Hamiltonian condition $i_{\widetilde{X}}\omega_H = 0$, i.e., $i_{\widetilde{X}}\omega = (i_{\widetilde{X}} dH)dt - dH$, means then $i_X\omega = -d_M H$ and $i_{\widetilde{X}} dH (= \widetilde{X}.H) = H_t$; the second relation can also be written as $X.H = 0$ or $d_M H(X) = 0$ and is a consequence of the first, since ω is skewsymmetric, and so $-d_M H(X) = i_X\omega(X) = \omega(X, X) = 0$. Since ω is nondegenerate, the relation $i_X\omega = -d_M H$ shows that the Hamiltonian field $\widetilde{X}$ exists and is unique. For the case $\omega = \sum dp_i \wedge dq_i$ the relation $i_X\omega = -d_M H$ amounts to the canonical equations $\dot{q}_i = H_{p_i}$, $\dot{p}_i = -H_{q_i}$.

We now construct the map F of the proposition: as noted after the proposition, it simply sends each line $x \times \mathbb{R}$ to the trajectory of $\widetilde{X}$ through $(x, 0)$. (In particular, we have $x = F(x, 0)$.) This is a diffeomorphism by standard theorems about ordinary differential equations. Clearly the vector fields ∂_t on $M \times \mathbb{R}$ map to $\widetilde{X}$ under F. It follows that i_{∂_t} nullifies $F^*\omega_H$.

We write $F^*\omega_H$ as $\omega_0 + \beta \wedge dt$, where ω_0 and β are nullified by ∂_t, i.e., do not involve any dt. The relation $i_{\partial_t} F^*\omega_H = 0$ then says $\beta = 0$; so we have $F^*\omega_H = \omega_0$. Since ω_H is closed, so is ω_0; the equation $d\omega_0 = 0$ implies that the t-derivatives of the coefficients a_{ij} of $\omega_0 = \sum a_{ij} dx_i \wedge dx_j$ vanish and that the form ω_0 does not depend on t (for this the domain of definition should be convex in the t-direction and connected). Thus $F^*\omega_H$ is simply a two-form on M, pulled back to $M \times \mathbb{R}$; and finally, since the map F is the identity on the slice $t = 0$, $F^*\omega_H$ equals ω. □

References

1. C. Carathéodory, *Variartionsrechnung und partielle Differentialgleichungen erster Ordnung*, Teubner, Leipzig, 1935, p. 124; English transl., *Calculus of variations and partial differential equations of first order*, Holden-Day, San Francisco, 1965, p. 125.
2. J. Moser, *On the volume elements of a manifold*, Trans. Amer. Math. Soc. **120** (1965), 286–294.
3. H. Samelson, *On Darboux's lemma*, Proc. Amer. Math. Soc. **70** (1978), 126–128.
4. A. Weinstein, *Symplectic structures on Banach manifolds*, Bull. Amer. Math. Soc. **75** (1969), 1040–1041.
5. E. T. Whittaker, *A treatise on the analytical dynamics of particles and rigid bodies*, Dover, New York, 1944, p. 317.

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