NEW THETA CONSTANT IDENTITIES II

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ABSTRACT. We apply the residue theorem to prove some of Ramanujan's identities and modular equations. Some of the identities already appeared in Israel J. Math. (**82** (1993), 133–141), but the proofs are given in this note.

1. INTRODUCTION

Theta constant identities are a classical object of study. They arise in relation to problems in combinatorics and, as such, are related to number-theoretic questions. They also arise in relation to moduli questions for Riemann surfaces; in this connection they are sometimes referred to as modular equations.

Classically theta functions and theta constants were studied for the case of integer characteristics, although there was some work done for the case of rational characteristics [M]. Recently, however [FK], it was shown that meromorphic modular forms for the principal congruence subgroups of the modular group $PSL(2, \mathbb{Z})$ are intimately related to theta constants with rational characteristics. This led to a generalization of the classical λ-function to the case of the principal congruence subgroup of level three (the classical λ function deals essentially with the level two case) and a new cubic theta constant identity. The identity, as do many identities in this theory, was derived from considerations of dimension of certain vector spaces. This in fact is the classical way of deriving theta function and theta constant identities [RF]. The results of [FK] raised the question of deriving similar identities related to *all* congruence subgroups. This question was dealt with in [FKO] where the cubic identity derived in [FK] was generalized to a pth power identity for all primes p greater than 3. There was also given without proof two additional identities which follow from similar considerations.

In this note we give the proofs of the two identities just mentioned. In addition, we derive some modular equations of Ramanujan. The idea is that working with theta constants with rational characteristics allows us to simplify the Ramanujan identities.

The underlying principle of our work is that every elliptic function gives rise to an identity among theta constants via an application of the residue theorem and the well-known fact that the sum of the residues of an elliptic function vanishes in the period parallelogram. This is the basic scientific principle. The art is in choosing the elliptic function in such a way that the classical identities of Jacobi, Riemann, and Ramanujan follow as consequences. In our work, or at least in the identities we derive in this paper, the idea is to choose the elliptic functions in such a way that after applying the residue theorem we are able to eliminate the denominators as a common factor.

This note is written as a sequel to [FKO] and thus should be read after [FKO]. It is also advisable to have looked at [FK] for more references to the literature and more background and motivation.

2. PROOF OF IDENTITIES III, IV OF [FKO]

We begin with the following remark:
Let

$$f(z) = \theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (7z, 7\tau).$$

$f(z)$ has simple zeros at the points $\{0, \frac{1}{7}, \frac{2}{7}, ..., \frac{6}{7}\}$ and therefore

$$f(z) = c(\tau) \prod_{l=1}^{l=7} \theta \begin{bmatrix} 1 \\ \frac{2l-1}{7} \end{bmatrix} (z, \tau),$$

where $c(\tau)$ is a constant which depends on τ which will not be necessary for us to compute (since it cancels out in the end). We use the above formula to compute the value of $f(z)$ at various points. The points at which we choose to compute are $\{\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$.

Formula (1) in [FKO] allows us to compute these quantities and the results are the following

Proposition 2.1.

$$f(\tfrac{1}{2}) = \theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (\tfrac{7}{2}, 7\tau) = \theta \begin{bmatrix} 1 \\ 8 \end{bmatrix} (0, 7\tau) = c(\tau) \prod_{l=1}^{l=7} \theta \begin{bmatrix} 1 \\ \frac{2l+6}{7} \end{bmatrix} (0, \tau),$$

$$f(\tfrac{\tau}{2}) = \theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (\tfrac{7\tau}{2}, 7\tau) = \exp(2\pi i[-\tfrac{7\tau}{8} - \tfrac{1}{4}])\theta \begin{bmatrix} 2 \\ 1 \end{bmatrix} (0, 7\tau)$$

$$= c(\tau) \prod_{l=1}^{l=7} \exp(2\pi i[-\tfrac{\tau}{8} - \tfrac{1}{4}\tfrac{2l-1}{7}])\theta \begin{bmatrix} 2 \\ \frac{2l-1}{7} \end{bmatrix} (0, \tau),$$

$$f(\tfrac{1+\tau}{2}) = \theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (\tfrac{7\tau+7}{2}, 7\tau) = \exp(2\pi i[-\tfrac{7\tau}{8} - 2])\theta \begin{bmatrix} 2 \\ 8 \end{bmatrix} (0, 7\tau)$$

$$= c(\tau) \prod_{l=1}^{l=7} \exp(2\pi i[-\tfrac{\tau}{8} - \tfrac{1}{4}\tfrac{2l+6}{7}])\theta \begin{bmatrix} 2 \\ \frac{2l+6}{7} \end{bmatrix} (0, \tau).$$

It follows from the above proposition that we have the following equalities:

$$(\text{A}) \quad \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 7\tau) = c(\tau) \prod_{l=1}^{l=7} \theta \begin{bmatrix} 1 \\ \frac{2l+6}{7} \end{bmatrix} (0, \tau),$$

$$(\text{B}) \quad \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 7\tau) = c(\tau) \prod_{l=1}^{l=7} \exp(2\pi i[-\frac{2l+6}{28}]) \theta \begin{bmatrix} 0 \\ \frac{2l+6}{7} \end{bmatrix} (0, \tau),$$

$$(\text{C}) \quad \exp(-\frac{\pi i}{2}) \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, 7\tau) = c(\tau) \prod_{l=1}^{l=7} \exp(2\pi i[-\frac{2l-1}{28}]) \theta \begin{bmatrix} 0 \\ \frac{2l-1}{7} \end{bmatrix} (0, \tau).$$

The above equalities all follow from equation 1) in [FKO].

We now make the crucial observation that the terms on the right-hand side of the equalities simplify considerably. It follows from equations 2) and 3) of [FKO] that we have

$$\theta \begin{bmatrix} 1 \\ \frac{2(8-l)+6}{7} \end{bmatrix} = \theta \begin{bmatrix} -1 \\ \frac{2l-22}{7} \end{bmatrix} = \theta \begin{bmatrix} 1-2 \\ \frac{2l+6}{7} - 4 \end{bmatrix} = \theta \begin{bmatrix} 1 \\ \frac{2l+6}{7} \end{bmatrix}$$

and that

$$\theta \begin{bmatrix} 0 \\ \frac{2(8-l)-1}{7} \end{bmatrix} = \theta \begin{bmatrix} 0 \\ \frac{2l-15}{7} \end{bmatrix} = \theta \begin{bmatrix} 0 \\ \frac{2l-1}{7} - 2 \end{bmatrix} = \theta \begin{bmatrix} 0 \\ \frac{2l-1}{7} \end{bmatrix}.$$

It thus follows that we can rewrite equations (A), (B) and (C) above as follows. (A) can be rewritten as

$$(\text{A}') \quad \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 7\tau) = -c(\tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau) \prod_{l=1}^{l=3} \theta^2 \begin{bmatrix} 1 \\ \frac{2l+6}{7} \end{bmatrix} (0, \tau),$$

(B) can be rewritten as

$$(\text{B}') \quad \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 7\tau) = -c(\tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) \prod_{l=1}^{l=3} \theta^2 \begin{bmatrix} 0 \\ \frac{2l+6}{7} \end{bmatrix} (0, \tau),$$

and finally (C) can be rewritten as

$$(\text{C}') \quad \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, 7\tau) = -c(\tau) \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, \tau) \prod_{l=1}^{l=3} \theta^2 \begin{bmatrix} 0 \\ \frac{2l-1}{7} \end{bmatrix} (0, \tau).$$

In particular we see that multiplication of the left sides of (A'), (B') and (C') by $\theta[\begin{smallmatrix}1\\0\end{smallmatrix}](0, \tau)$, $\theta[\begin{smallmatrix}0\\0\end{smallmatrix}](0, \tau)$ and $\theta[\begin{smallmatrix}0\\1\end{smallmatrix}](0, \tau)$ respectively makes the right-hand sides products of squares. This is our main observation. In order to get the identity for which we are looking we have to write these expressions out more fully and see what the residue theorem has to contribute. We begin by considering the following function.

Let

$$f(z) = \frac{\theta \begin{bmatrix} 1 \\ \frac{5}{7} \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ \frac{3}{7} \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ \frac{13}{7} \end{bmatrix} (z, \tau)}{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z, \tau)}.$$

We see that $f(z)$ is a doubly periodic function with periods 1 and τ. This follows from the usual properties of the theta function. Hence, the sum of the residues of this function in a period parallelogram must vanish. The poles of this function are at the points $\{\frac{1+\tau}{2}, \frac{1}{2}, \frac{\tau}{2}\}$, and therefore we must compute the sum of the residues at these three points.

Lemma 2.2.

$$\operatorname{Res}_{z=\frac{1}{2}} f(z) = \frac{\theta\begin{bmatrix}1\\\frac{1}{12}\end{bmatrix}\theta\begin{bmatrix}1\\\frac{1}{10}\end{bmatrix}\theta\begin{bmatrix}1\\\frac{1}{20}\end{bmatrix}}{\theta'\begin{bmatrix}1\\1\end{bmatrix}\theta\begin{bmatrix}0\\0\end{bmatrix}\theta\begin{bmatrix}0\\1\end{bmatrix}},$$

$$\operatorname{Res}_{z=\frac{\tau}{2}} f(z) = \frac{\theta\begin{bmatrix}0\\\frac{5}{7}\end{bmatrix}\theta\begin{bmatrix}0\\\frac{3}{7}\end{bmatrix}\theta\begin{bmatrix}0\\\frac{13}{7}\end{bmatrix}}{\theta'\begin{bmatrix}1\\1\end{bmatrix}\theta\begin{bmatrix}1\\0\end{bmatrix}\theta\begin{bmatrix}0\\0\end{bmatrix}},$$

$$\operatorname{Res}_{z=\frac{\tau+1}{2}} f(z) = -\frac{\theta\begin{bmatrix}0\\\frac{12}{7}\end{bmatrix}\theta\begin{bmatrix}0\\\frac{10}{7}\end{bmatrix}\theta\begin{bmatrix}0\\\frac{20}{7}\end{bmatrix}}{\theta'\begin{bmatrix}1\\1\end{bmatrix}\theta\begin{bmatrix}1\\0\end{bmatrix}\theta\begin{bmatrix}0\\1\end{bmatrix}}.$$

Proof. The residue of the function is computed in the usual fashion using equation 1) of [FKO]. □

Theorem 2.3. *There is a choice of square root so that*

$$\sqrt{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,7\tau)},$$

$$= \sqrt{\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(0,7\tau)} + \sqrt{\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,7\tau)}.$$

Proof. The fact that the sum of the residues of an elliptic function vanishes in the period parallelogram gives us the identity

$$\operatorname{Res}_{z=\frac{1}{2}} f(z) + \operatorname{Res}_{z=\frac{\tau}{2}} f(z) + \operatorname{Res}_{z=\frac{\tau+1}{2}} f(z) = 0.$$

The lemma above translates this identity to

$$\theta\begin{bmatrix}0\\0\end{bmatrix}\theta\begin{bmatrix}0\\\frac{20}{7}\end{bmatrix}\theta\begin{bmatrix}0\\\frac{10}{7}\end{bmatrix}\theta\begin{bmatrix}0\\\frac{12}{7}\end{bmatrix} - \theta\begin{bmatrix}0\\1\end{bmatrix}\theta\begin{bmatrix}0\\\frac{13}{7}\end{bmatrix}\theta\begin{bmatrix}0\\\frac{3}{7}\end{bmatrix}\theta\begin{bmatrix}0\\\frac{5}{7}\end{bmatrix},$$

$$- \theta\begin{bmatrix}1\\0\end{bmatrix}\theta\begin{bmatrix}1\\\frac{20}{7}\end{bmatrix}\theta\begin{bmatrix}1\\\frac{10}{7}\end{bmatrix}\theta\begin{bmatrix}1\\\frac{12}{7}\end{bmatrix} = 0.$$

We now multiply equations (A'), (B'), and (C') as suggested immediately after the equations, extract the square roots, and obtain the statement of the theorem. □

We now turn our attention to the identity IV in [FKO]. The derivation is very similar to the derivation of the theorem above. We begin with the following remark parallel to the remark made at the beginning of this paper:

Let $f(z) = \theta[^1_1](9z, 9\tau)/\theta[^1_1](z, \tau)$. $f(z)$ has simple zeros at the points $\{\frac{1}{9}, \frac{2}{9}, \ldots, \frac{8}{9}\}$, and therefore

$$f(z) = c(\tau) \frac{\prod_{l=1}^{l=9} \theta \begin{bmatrix} 1 \\ \frac{2l-1}{9} \end{bmatrix} (z, \tau)}{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z, \tau)}.$$

We compute the value of this function which is holomorphic in the entire plane at the points $\{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$.

Proposition 2.4.

$$f(0) = 9 \frac{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} (0, 9\tau)}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} (0, \tau)} = c(\tau) \prod_{l=1}^{l=4} \theta^2 \begin{bmatrix} 1 \\ \frac{2l-1}{9} \end{bmatrix} (0, \tau),$$

$$f(\frac{1}{2}) = \frac{\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 9\tau)}{\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau)} = c(\tau) \prod_{l=1}^{l=4} \theta^2 \begin{bmatrix} 1 \\ \frac{2l+8}{9} \end{bmatrix} (0, \tau),$$

$$f(\frac{\tau}{2}) = \exp(-2\pi i \tau) \frac{\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, 9\tau)}{\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, \tau)} = \exp(-2\pi i \tau) c(\tau) \prod_{l=1}^{l=4} \theta^2 \begin{bmatrix} 0 \\ \frac{2l-1}{9} \end{bmatrix} (0, \tau),$$

$$f(\frac{1+\tau}{2}) = \exp(-2\pi i \tau) \frac{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 9\tau)}{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau)} = \exp(-2\pi i \tau) c(\tau) \prod_{l=1}^{l=4} \theta^2 \begin{bmatrix} 0 \\ \frac{2l+8}{9} \end{bmatrix} (0, \tau).$$

Proof. The proof is the same as for the previous proposition. □

Let

$$f(z) = \frac{\theta \begin{bmatrix} 1 \\ \frac{1}{9} \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ \frac{7}{9} \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ \frac{13}{9} \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ \frac{15}{9} \end{bmatrix} (z, \tau)}{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \tau)}.$$

It is clear that $f(z)$ is a doubly periodic function with periods 1 and τ, and therefore the sum of the residues of this function in a period parallelogram is zero. As in the case of the previous lemma we see that:

$$\text{Res}_{z=0} f(z) = \frac{\theta \begin{bmatrix} 1 \\ \frac{1}{9} \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 1 \\ \frac{7}{9} \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 1 \\ \frac{13}{9} \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 1 \\ \frac{15}{9} \end{bmatrix} (0, \tau)}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau)},$$

$$\text{Res}_{z=\frac{1}{2}} f(z) = -\frac{\theta\begin{bmatrix}1\\\frac{10}{9}\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\\frac{16}{9}\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\\frac{16}{9}\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\\frac{24}{9}\end{bmatrix}(0,\tau)}{\theta'\begin{bmatrix}1\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)},$$

$$\text{Res}_{z=\frac{1}{2}} f(z) = -\frac{\theta\begin{bmatrix}0\\\frac{1}{9}\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\\frac{7}{9}\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\\frac{13}{9}\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\\frac{15}{9}\end{bmatrix}(0,\tau)}{\theta'\begin{bmatrix}1\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)},$$

$$\text{Res}_{z=\frac{1+\tau}{2}} f(z) = \frac{\theta\begin{bmatrix}0\\\frac{10}{9}\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\\frac{16}{9}\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\\frac{22}{9}\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\\frac{24}{9}\end{bmatrix}(0,\tau)}{\theta'\begin{bmatrix}1\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)}.$$

Theorem 2.5. *There is a choice of square root so that*

$$\sqrt{\frac{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,9\tau)}{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)}} - \sqrt{\frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(0,9\tau)}{\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)}} - \sqrt{\frac{\theta\begin{bmatrix}1\\0\end{bmatrix}(0,9\tau)}{\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)}}$$

$$= -3\sqrt{\frac{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,9\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(0,9\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,9\tau)}{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)}}.$$

Proof. The proof is an immediate consequence of the fact that the sum of the residues must vanish and the fact that these residues are in fact the square roots of the values of the function previously computed. □

This concludes the proofs of Identities III, IV of [FKO]. If the method were limited to obtaining proofs of the above identities, it would in the authors' opinion merely be an interesting curiosity; however, the method seems to be very far ranging and useful. In order to show the usefulness and utility of the method we use the technique to obtain other classical identities:

Consider the following function which the reader can clearly check to see that it is a doubly periodic function with periods 1 and τ. The function is

$$f(z) = \frac{\theta\begin{bmatrix}a\\b\end{bmatrix}(z,\tau)\theta\begin{bmatrix}-a\\-b\end{bmatrix}(z,\tau)\theta\begin{bmatrix}c\\d\end{bmatrix}(z,\tau)\theta\begin{bmatrix}-c\\-d\end{bmatrix}(z,\tau)}{\theta\begin{bmatrix}0\\0\end{bmatrix}(z,\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(z,\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(z,\tau)\theta\begin{bmatrix}1\\1\end{bmatrix}(z,\tau)}.$$

In this function the vectors (a, b) and (c, d) are arbitrary vectors in R^2. The

residue theorem applied to this function yields:

$$\theta^2 \begin{bmatrix} a \\ b \end{bmatrix} \theta^2 \begin{bmatrix} c \\ d \end{bmatrix} - \theta \begin{bmatrix} a+1 \\ b \end{bmatrix} \theta \begin{bmatrix} -a+1 \\ -b \end{bmatrix} \theta \begin{bmatrix} c+1 \\ d \end{bmatrix} \theta \begin{bmatrix} -c+1 \\ -d \end{bmatrix}$$

$$- \theta \begin{bmatrix} a \\ b+1 \end{bmatrix} \theta \begin{bmatrix} -a \\ -b+1 \end{bmatrix} \theta \begin{bmatrix} c \\ c+1 \end{bmatrix} \theta \begin{bmatrix} -c \\ -c+1 \end{bmatrix}$$

$$+ \theta \begin{bmatrix} a+1 \\ b+1 \end{bmatrix} \theta \begin{bmatrix} -a+1 \\ -b+1 \end{bmatrix} \theta \begin{bmatrix} c+1 \\ d+1 \end{bmatrix} \theta \begin{bmatrix} -c+1 \\ -d+1 \end{bmatrix} = 0.$$

Several points are already worthy of note. If we set the vectors $(a, b) = (c, d) = (0, 0)$, we obtain the classical Jacobi identity

$$\theta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \theta^4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \theta^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$$

At the other extreme, we can set $z = a\frac{\tau}{2} + \frac{b}{2}$ and $w = c\frac{\tau}{2} + \frac{d}{2}$ and rewrite the above identity as

$$\theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \tau) \theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (w, \tau) - \theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z, \tau) \theta^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (w, \tau)$$

$$- \theta^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z, \tau) \theta^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (w, \tau) + \theta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z, \tau) \theta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (w, \tau) = 0.$$

By choosing different values for z, w we can obtain many identities.

3. MODULAR EQUATIONS AND THE RESIDUE THEOREM

We shall concentrate on identities involving the cubic modular equation which are related to identity I in [FKO]. We begin by recalling the following definitions:

$$\lambda(\tau) = \frac{\theta^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau)}{\theta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau)} \text{ and } m(\tau) = \frac{\theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau)}{\theta^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 3\tau)}.$$

$\lambda(\tau)$ is of course the classical λ function and $m(\tau)$ is a multiplier. We recall that in [FKO] we derived the equation

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 3\tau)$$

$$= \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, 3\tau) + \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 3\tau).$$

This identity followed immediately from the residue theorem by considering the function

$$\frac{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (3z, 3\tau)}{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z, \tau)}.$$

In order to obtain similar identities, we consider in place of this function the functions:

$$g(z) = \frac{\theta \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} (3z, 3\tau)}{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z, \tau)},$$

$$h(z) = \frac{\theta^3 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} (z, \tau)}{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z, \tau)}.$$

The first of these leads to the identity

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix} (0, 3\tau)$$

$$= \exp(\frac{2\pi i}{3}) \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} (0, 3\tau) + \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} (0, 3\tau),$$

and the second leads to the identity

(A'')
$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) \theta^3 \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix} (0, \tau) - \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, \tau) \theta^3 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} (0, \tau)$$

$$= \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau) \theta^3 \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} (0, \tau).$$

It is interesting to observe that the first of these equations is in fact derivable algebraically from equation I of [FKO], the cubic modular equation. If we take the cubic modular equation and multiply the variable τ by 3, we obtain the same equation with the variables 3τ and 9τ. If we subtract one from the other we obtain the following equation:

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 3\tau)(\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) - \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, 9\tau))$$

$$= \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, 3\tau)(\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, \tau) - \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, 9\tau))$$

$$+ \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 3\tau)(\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau) - \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, 9\tau)).$$

We now need the following lemma which must be known, but again we do not have a reference for it.

Lemma 3.1. *For any* $n > 1$

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z, \tau) = \sum_{l=0}^{l=n-1} \theta \begin{bmatrix} \frac{2l+\epsilon}{n} \\ n\epsilon' \end{bmatrix} (nz, n^2\tau).$$

Proof. The proof is simply writing the definition of the left-hand side and summing over the separate residue classes of the integers modulo n. □

In particular, for $n = 3$, we obtain the following:

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) = \sum_{l=0}^{l=2} \theta \begin{bmatrix} \frac{2l}{3} \\ 0 \end{bmatrix} (0, 9\tau),$$

$$\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, \tau) = \sum_{l=0}^{l=2} \theta \begin{bmatrix} \frac{2l}{3} \\ 3 \end{bmatrix} (0, 9\tau),$$

$$\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau) = \sum_{l=0}^{l=2} \theta \begin{bmatrix} \frac{2l+3}{3} \\ 0 \end{bmatrix} (0, 9\tau).$$

The fact that $\theta[\frac{2}{0}] = \theta[\frac{4}{0}]$ and the fact that $\theta[\frac{2}{3}] = \theta[\frac{4}{3}] = \exp(\frac{2\pi i}{3})\theta[\frac{2}{1}]$, and finally that $\theta[\frac{3}{0}] = \theta[\frac{7}{0}] = \theta[\frac{1}{0}]$ completes the proof, once we observe that the resulting equation in the variables 3τ and 9τ can be changed to τ and 3τ.

We now show how identity (A″) is actually entry (vi) in [B, p. 230]. Let us define a function

$$f(z) = \frac{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z, \tau/3)}{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z, \tau) \theta \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} (z, \tau) \theta \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix} (z, \tau)}.$$

This function, as the reader can easily check, is an elliptic function with periods 1 and τ. It is holomorphic and so constant. The constant is however a function of τ. We evaluate this function at the points $z = 1/2$, $z = \tau/2$ and at the point $z = \frac{1+\tau}{2}$.
We find:

$$f(\frac{1}{2}) = \frac{\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \frac{\tau}{3})}{\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} \frac{5}{3} \\ 0 \end{bmatrix} (0, \tau)} = \frac{\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \frac{\tau}{3})}{\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \tau) \theta^2 \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} (0, \tau)},$$

$$f(\frac{\tau}{2}) = \frac{\theta \begin{bmatrix} 4 \\ 1 \end{bmatrix} (0, \frac{\tau}{3})}{\theta \begin{bmatrix} 2 \\ 1 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} \frac{4}{3} \\ 0 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} \frac{8}{3} \\ 1 \end{bmatrix} (0, \tau)}$$

$$= \exp(\frac{2\pi i}{3}) \frac{\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, \frac{\tau}{3})}{\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0, \tau) \theta^2 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} (0, \tau)},$$

$$f(\frac{1+\tau}{2}) = \frac{\theta \begin{bmatrix} 4 \\ 2 \end{bmatrix} (0, \frac{\tau}{3})}{\theta \begin{bmatrix} 2 \\ 2 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} \frac{4}{3} \\ 2 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} \frac{8}{3} \\ 2 \end{bmatrix} (0, \tau)} = \frac{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \frac{\tau}{3})}{\theta^2 \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix} (0, \tau) \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau)}.$$

The fact that $f(\frac{1}{2}) = f(\frac{\tau}{2}) = f(\frac{1+\tau}{2})$ allows us to express the rational characteristics appearing in equation (A'') in terms of the integer characteristics. If we then use the definition of λ as given above and the quartic theta identity of Jacobi, we get

$$\left(\frac{\lambda^3(\tau)}{\lambda(3\tau)}\right)^{1/8} - \left(\frac{(1 - \lambda(\tau))^3}{(1 - \lambda(3\tau))}\right)^{1/8} = 1.$$

This is Ramanujan's identity referred to above.

In order to obtain another identity of this type, we now consider the three functions

$$f_1(z) = \frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(3z, 3\tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(z, \tau)}{\theta^2\begin{bmatrix}1\\1\end{bmatrix}(z, \tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(z, \tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(z, \tau)},$$

$$f_2(z) = \frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(3z, 3\tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(z, \tau)}{\theta^2\begin{bmatrix}1\\1\end{bmatrix}(z, \tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(z, \tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(z, \tau)},$$

$$f_3(z) = \frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(3z, 3\tau)\theta\begin{bmatrix}0\\1\end{bmatrix}(z, \tau)}{\theta^2\begin{bmatrix}1\\1\end{bmatrix}(z, \tau)\theta\begin{bmatrix}0\\0\end{bmatrix}(z, \tau)\theta\begin{bmatrix}1\\0\end{bmatrix}(z, \tau)}.$$

Each of the functions $f_i(z)$ is an elliptic function with periods 1 and τ and has three simple poles in the period parallelogram. We apply the residue theorem and obtain the following:

$$\frac{\theta\begin{bmatrix}1\\0\end{bmatrix}(0, 3\tau)}{\theta\begin{bmatrix}1\\0\end{bmatrix}(0, \tau)}\theta^2\begin{bmatrix}0\\1\end{bmatrix}(0, \tau) - \frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(0, 3\tau)}{\theta\begin{bmatrix}0\\1\end{bmatrix}(0, \tau)}\theta^2\begin{bmatrix}1\\0\end{bmatrix}(0, \tau)$$

$$= -3\frac{\theta'\begin{bmatrix}1\\1\end{bmatrix}(0, 3\tau)}{\theta'\begin{bmatrix}1\\1\end{bmatrix}(0, \tau)}\theta^2\begin{bmatrix}0\\0\end{bmatrix}(0, \tau).$$

This identity comes from the function $f_1(z)$. Considering $f_2(z)$ and $f_3(z)$ leads to the following:

$$\frac{\theta\begin{bmatrix}0\\0\end{bmatrix}(0, 3\tau)}{\theta\begin{bmatrix}0\\0\end{bmatrix}(0, \tau)}\theta^2\begin{bmatrix}0\\1\end{bmatrix}(0, \tau) - \frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(0, 3\tau)}{\theta\begin{bmatrix}0\\1\end{bmatrix}(0, \tau)}\theta^2\begin{bmatrix}0\\0\end{bmatrix}(0, \tau)$$

$$= -3\frac{\theta'\begin{bmatrix}1\\1\end{bmatrix}(0, 3\tau)}{\theta'\begin{bmatrix}1\\1\end{bmatrix}(0, \tau)}\theta^2\begin{bmatrix}1\\0\end{bmatrix}(0, \tau),$$

$$\frac{\theta\begin{bmatrix}1\\0\end{bmatrix}(0,3\tau)}{\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)}\theta^2\begin{bmatrix}0\\0\end{bmatrix}(0,\tau) - \frac{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,3\tau)}{\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)}\theta^2\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)$$

$$= -3\frac{\theta'\begin{bmatrix}1\\1\end{bmatrix}(0,3\tau)}{\theta'\begin{bmatrix}1\\1\end{bmatrix}(0,\tau)}\theta^2\begin{bmatrix}0\\1\end{bmatrix}(0,\tau).$$

These equations can be combined to give the cubic modular identity. In fact, dividing the first equation by $\theta^2[\begin{smallmatrix}0\\0\end{smallmatrix}](0,\tau)$ and the second equation by $\theta^2[\begin{smallmatrix}1\\0\end{smallmatrix}](0,\tau)$ and equating the resulting expressions together with the classical quartic gives the cubic modular equation we derived in [FKO]. This seems to suggest that these equations are more fundamental than the classical cubic modular identity. In fact, if we use the first of the above three equations, and use the definitions of $m(\tau)$ and $\lambda(\tau)$ given above, we can rewrite the identity as

$$\frac{3}{m} = \left(\frac{\lambda(\tau)^3}{\lambda(3\tau)}\right)^{1/4} - \left(\frac{(1-\lambda(\tau))^3}{1-\lambda(3\tau)}\right)^{1/4}.$$

Dividing the last identity by (A″) we get

$$\frac{3}{m} = \left(\frac{\lambda(\tau)^3}{\lambda(3\tau)}\right)^{1/8} + \left(\frac{(1-\lambda(\tau))^3}{1-\lambda(3\tau)}\right)^{1/8}$$

which appears as Entry 5 (iii) in [B, p.230].

Another immediate consequence of the preceding calculation is the following: The equality $f(\frac{1}{2}) = f(\frac{1+\tau}{2})$ yields

$$\lambda\left(\frac{\tau}{3}\right) = \lambda(\tau)\frac{\theta^8\begin{bmatrix}\frac{1}{3}\\0\end{bmatrix}(0,\tau)}{\theta^8\begin{bmatrix}\frac{2}{3}\\0\end{bmatrix}(0,\tau)}.$$

Similarly, the equality $f(\frac{\tau}{2}) = f(\frac{1+\tau}{2})$ yields

$$\exp\left(\frac{2\pi i}{3}\right)\left(1 - \lambda\left(\frac{\tau}{3}\right)\right) = (1 - \lambda(\tau))\frac{\theta^8\begin{bmatrix}\frac{2}{3}\\1\end{bmatrix}(0,\tau)}{\theta^8\begin{bmatrix}\frac{2}{3}\\0\end{bmatrix}(0,\tau)}.$$

These two identities lead to the following expression for the classical λ function in terms of these rational characteristics:

$$\lambda(\tau) = \frac{\exp(\frac{2\pi i}{3})\theta^8\begin{bmatrix}\frac{2}{3}\\0\end{bmatrix}(0,\tau) - \theta^8\begin{bmatrix}\frac{2}{3}\\1\end{bmatrix}(0,\tau)}{\exp(\frac{2\pi i}{3})\theta^8\begin{bmatrix}\frac{1}{3}\\0\end{bmatrix}(0,\tau) - \theta^8\begin{bmatrix}\frac{2}{3}\\1\end{bmatrix}(0,\tau)}.$$

It is an immediate consequence that the following proportionalities hold:

Theorem 3.2.

$$
\frac{\exp(\frac{2\pi i}{3})\theta^8 \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}(0,\tau) - \theta^8 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}(0,\tau)}{\theta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix}(0,\tau)}
$$

$$
= \frac{\exp(\frac{2\pi i}{3})\theta^8 \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix}(0,\tau) - \theta^8 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}(0,\tau)}{\theta^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,\tau)}
$$

$$
= \exp(\frac{2\pi i}{3})\left(\frac{\theta^8 \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}(0,\tau) - \theta^8 \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix}(0,\tau)}{\theta^4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}(0,\tau)} \right).
$$

4. CONCLUSION

This note began with the purpose of giving the proofs of identities III and IV of [FKO], and along the way we have tried to convince the reader that the technique of the residue theorem together with some calculation is enough to obtain some remarkable identities among theta constants. In particular, we have obtained what we believe are new proofs for some of Ramanujan's identities. We have not written down all the identities we can obtain in this manner nor have we been able to obtain all identities that are known in this way. We believe that this technique can be useful in *understanding* the identities as well. The lesson we learn from the technique is that complicated irrationalities which appear in the Ramanujan identities disappear when you are in the realm of noninteger characteristics. Some of the ideas of the current paper and [FKO] have been generalized by the second author to the case of the several variable theta constants and will be appearing in his Hebrew University dissertation.

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