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LINEAR MAPPINGS THAT PRESERVE POTENT OPERATORS

MATJAŽ OMLADIČ AND PETER ŠEMRL

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ABSTRACT. Let H and K be a complex Hilbert spaces, while $\mathscr{B}(H)$ and $\mathscr{B}(K)$ denote the algebras of all linear bounded operators on H and K, respectively. We characterize surjective linear mappings from $\mathscr{B}(H)$ onto $\mathscr{B}(K)$ that preserve potent operators in both directions.

The problem of characterizing linear mappings ϕ on the algebra M_n of all $n \times n$ matrices which preserve some subsets Γ of M_n (that is, $\phi(\Gamma) \subset \Gamma$) has attracted the attention of many mathematicians in the last few decades [12]. Let us mention here some examples of such subsets: the case when Γ is the set of all singular matrices was considered by Dieudonné [7], Γ is a linear group by Dixon [8], and Γ is the set of all nilpotent matrices by Botta, Pierce, and Watkins [1]. In a recent paper [2], motivated by a problem of characterizing local automorphisms and local derivations of some operator algebras (see, e.g., [11]), Brešar and Šemrl considered the case when Γ is the set of all idempotents in M_n. The same authors also considered a more general situation [3], namely, they characterized linear transformations preserving potent matrices (recall that a matrix A is said to be potent if $A^r = A$ for some integer $r \geq 2$) as well as linear mappings that preserve the set of all r-potent matrices $\Pi_r = \{A \in M_n : A^r = A\}$ for some integer $r \geq 2$.

In recent years there has also been considerable interest in linear preserver problems on operator algebras over infinite-dimensional spaces [2, 4, 5, 6, 9, 13, 14]. It is the aim of this note to continue this work by studying linear mappings $\phi: \mathscr{B}(H) \to \mathscr{B}(K)$ that preserve potent operators in both directions. Here, H and K are nontrivial complex Hilbert spaces, while $\mathscr{B}(H)$ and $\mathscr{B}(K)$ denote the algebras of all bounded linear operators on H and K, respectively. The main idea is different from the one used in the finite-dimensional case [3]. We also need a stronger assumption on ϕ: it must preserve potent operators in both directions, that is, for every $A \in \mathscr{B}(H)$ the operator $\phi(A)$ is a potent operator if and only if the same is true for A.

Our proof is based on the following three results.

Theorem 1 [15]. *Let H be an infinite-dimensional Hilbert space. Then every operator $A \in \mathscr{B}(H)$ is a sum of five idempotents.*

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Theorem 2 [10]. *Let $\mathscr{A}$ be a Banach algebra, and let p, $q \in \mathscr{A}$ be idempotents. Then $p+q$ is an idempotent if and only if* $\sup_{n\in\mathbb{N}} \|(p+q)^n\| < \infty$.

Theorem 3 [4]. *Let H and K be Hilbert spaces, and let $\phi: \mathscr{B}(H) \to \mathscr{B}(K)$ be a bijective linear mapping. Assume that $\phi(P) \in \mathscr{B}(K)$ is an idempotent whenever $P \in \mathscr{B}(H)$ is an idempotent. Then there is a bounded bijective linear operator $V: H \to K$ such that either*

(i) $\phi(A) = VAV^{-1}$ *for every $A \in \mathscr{B}(H)$; or*

(ii) $\phi(A) = VA^{\mathrm{tr}}V^{-1}$ *for every $A \in \mathscr{B}(H)$, where A^{tr} denotes the transpose of A relative to a fixed but arbitrary orthonormal basis.*

This last result was proved in [4] only for the special case $H = K$. Almost the same proof works also in this more general setting.

Throughout the paper, for any vectors x, y we shall denote the scalar product of these two vectors by y^*x, while xy^* will denote the rank-one operator defined by $(xy^*)z = (y^*z)x$ for every vector z. Note that every operator of rank one can be written in this form. The operator xy^* is an idempotent if and only if $y^*x = 1$. Now we are ready to prove our result.

Theorem 4. *Let H and K be Hilbert spaces, and let $\phi: \mathscr{B}(H) \to \mathscr{B}(K)$ be a surjective linear mapping. Then the following conditions are equivalent:*

(i) *For every $A \in \mathscr{B}(H)$ the operator $\phi(A)$ is potent if and only if A is potent.*

(ii) *There exists an integer $r \geq 2$ such that for every $A \in \mathscr{B}(H)$ we have $(\phi(A))^r = \phi(A)$ if and only if $A^r = A$.*

(iii) *ϕ is either of the form $\phi(A) = cVAV^{-1}$ or $\phi(A) = cVA^{\mathrm{tr}}V^{-1}$, where $V: H \to K$ is a bounded bijective linear operator, $c \in \mathbb{C}$ is a root of unity, and A^{tr} denotes the transpose of A relative to any basis of H, fixed in advance.*

Proof. It is clear that (iii) implies (i) and (ii). We shall prove the converse implications. Let us first point out a simple observation which will be used later. An operator $A \in \mathscr{B}(H)$ is r-potent if and only if there exist nonzero idempotents $P_1, \dots, P_k$ and pairwise different $(r-1)$-roots of unity $\mu_1, \dots, \mu_k$ such that $A = \sum_{i=1}^k \mu_i P_i$ and $P_iP_j = 0$ for $i \neq j$. In order to see this we denote

$$\lambda_0 = 0 \qquad \text{and} \qquad \lambda_m = \exp\left(\frac{2m\pi i}{r-1}\right), \quad m = 1, \dots, r-1.$$

The polynomials $p_0, p_1, \dots, p_{r-1}$ defined by relations

$$p_i(\lambda) = \prod_{m=0,\, m\neq i}^{r-1} (\lambda - \lambda_m), \qquad \text{for } i = 0, 1, \dots, r-1,$$

have the greatest common divisor equal to 1, so that there exist polynomials $q_0, \dots, q_{r-1}$ with the property that $\sum_{i=0}^{r-1} p_i q_i = 1$. It follows that every x from H can be written as $x = \sum_{i=0}^{r-1} x_i$ where $x_i = p_i(A)q_i(A)x$, $i = 0, \dots, r-1$. Clearly, we have that $x_i \in \operatorname{Ker}(A - \lambda_i)$, $i = 0, 1, \dots, r-1$. Thus, we have proved that

$$H = \bigoplus_{i=0}^{r-1} H_i,$$

where $H_i = \operatorname{Ker}(A-\lambda_i)$, $i = 0, 1, \ldots, r-1$. It is easy to see that the nontrivial among the projections P_i of H onto H_i for $i = 1, \ldots, r-1$, associated with the above direct sum, have the desired property.

Assume now that condition (i) is satisfied. Then the mapping ϕ is injective, since the kernel of ϕ is a linear space which consists of potent operators only. This implies that ϕ is actually bijective.

Clearly, operator $S = \phi(I)$ is potent so that, by above, there exist a positive integer k, nonzero idempotents $Q_1, \ldots, Q_k \in \mathscr{B}(K)$, and pairwise different roots of unity $\mu_1, \ldots, \mu_k$ such that $S = \sum_{i=1}^k \mu_i Q_i$ and $Q_i Q_j = 0$ for $i \neq j$. We claim that $\sum_{i=1}^k Q_i = I$. Assume to the contrary that $Q = I - \sum_{i=1}^k Q_i$ is a nonzero idempotent. Then $S + cQ$ is a potent operator for every root of unity c. It follows that $\phi^{-1}(S + cQ) = I + c\phi^{-1}(Q)$ is a potent operator for every root of unity c, where $\phi^{-1}(Q) \neq 0$. This leads to a contradiction, thus showing that $\sum_{i=1}^k Q_i = I$.

Observe that $P_i = \mu_i \phi^{-1}(Q_i)$ are nonzero potents for $i = 1, \ldots, k$. Clearly, $S - \mu_i Q_i$ and $S - 2\mu_i Q_i$ are all potent operators. The same must therefore be true for operators $I - P_i$ and $I - 2P_i$, and this implies that P_i is an idempotent for every $i = 1, \ldots, k$. Obviously, we have that $I = \sum_{i=1}^k P_i$. Moreover, $\mu_i Q_i + \mu_j Q_j$ is a potent operator if $i \neq j$. By the assumption, this implies that $P_i + P_j$ is a potent operator. Using Theorem 2 we conclude that $P_i P_j = 0$ for $i \neq j$.

The algebra $\mathscr{A}_i = \{A \in \mathscr{B}(H) : P_i A P_i = A\}$, $i = 1, \ldots, k$, is isomorphic to $\mathscr{B}(H_i)$, where H_i denotes the image of the idempotent P_i. Similarly, for every $i = 1, \ldots, k$, the algebra $\mathscr{B}_i = \{A \in \mathscr{B}(K) : Q_i A Q_i = A\}$ is isomorphic to $\mathscr{B}(K_i)$ with $K_i = \operatorname{Im} Q_i$. Fix i for a while, and let P be an arbitrary idempotent in $\mathscr{A}_i$. If we denote $T = \sum_{j=1}^k \mu_j^{-1} P_j$, then $T - \mu_i^{-1} P$ and $T - 2\mu_i^{-1} P$ are potent operators in $\mathscr{B}(H)$. It follows that $I - \mu_i^{-1}\phi(P)$ and $I - 2\mu_i^{-1}\phi(P)$ are potent operators, which further forces $\mu_i^{-1}\phi(P)$ to be an idempotent. Since $\mu_i^{-1} P + \mu_j^{-1} P_j$ is a potent operator for every j different from i, the sum of idempotents $\mu_i^{-1}\phi(P) + Q_j$ is a potent operator, and consequently, by Theorem 2, we have that $\phi(P) Q_j = Q_j \phi(P) = 0$, or in other words, $\phi(P)$ belongs to $\mathscr{B}_i$. The linear span of all idempotents from $\mathscr{A}_i$ is the whole algebra $\mathscr{A}_i$. One can easily verify this fact in the case that H_i is finite dimensional, while in the infinite-dimensional case this statement follows from Theorem 1. This implies that ϕ maps $\mathscr{A}_i$ into $\mathscr{B}_i$. Similarly, we can prove that ϕ^{-1} maps $\mathscr{B}_i$ into $\mathscr{A}_i$, or in other words, the restriction of ϕ to the subalgebra $\mathscr{A}_i$ is a bijective mapping from $\mathscr{A}_i$ onto $\mathscr{B}_i$ for every $i = 1, \ldots, k$. Applying Theorem 3 we see, in particular, that $\mu_i^{-1}\phi$ maps every idempotent of rank one from $\mathscr{A}_i$ into an idempotent of rank one from $\mathscr{B}_i$. This is true, of course, for any index $i = 1, 2, \ldots, k$.

Next, we shall prove that $k = 1$ or, equivalently, that $\phi(I) = cI$ for some root of unity c. Assume to the contrary that $k > 1$. Let $xy^* \in \mathscr{A}_1$ and $zw^* \in \mathscr{A}_2$ be any idempotents of rank one. Then $y^*x = w^*z = 1$, and also, $(xy^*)(zw^*) = (zw^*)(xy^*) = 0$, and this implies that $y^*z = w^*x = 0$. Let us denote $\phi(xy^*) = \mu_1 x_1 y_1^*$ and $\phi(zw^*) = \mu_2 z_1 w_1^*$ to get that by the above $y_1^* x_1 = w_1^* z_1 = 1$ and $y_1^* z_1 = w_1^* x_1 = 0$.

For every complex number λ the operators $xy^* + \lambda xw^*$ and $zw^* + \lambda xw^*$

are idempotents. It follows that for every complex number λ there exists an integer $n_\lambda \geq 2$ such that

$$(x_1y_1^* + \lambda\phi(xw^*))^{n_\lambda} = x_1y_1^* + \lambda\phi(xw^*).$$

Clearly, there exists an integer $n_0 \geq 2$ such that the above relation with $n_0 = n_\lambda$ holds for infinitely many λ's, but then this must be fulfilled for every complex λ. Comparing the coefficients at λ we obtain

$$\phi(xw^*)x_1y_1^* + (n_0 - 2)x_1y_1^*\phi(xw^*)x_1y_1^* + x_1y_1^*\phi(xw^*) = \phi(xw^*).$$

Multiplying this relation from both sides by $x_1y_1^*$ we get that $x_1y_1^*\phi(xw^*)x_1y_1^* = 0$. Putting $u = \phi(xw^*)x_1$ and $v = \phi(xw^*)^*y_1$ we finally conclude that

$$\phi(xw^*) = x_1v^* + uy_1^*.$$

Similarly, we can see that there exist vectors u_1, $v_1 \in K$ such that

$$\phi(xw^*) = z_1v_1^* + u_1w_1^*.$$

Standard arguments show that the last two relations imply existence of complex numbers α and β such that

$$\phi(xw^*) = \alpha x_1w_1^* + \beta z_1y_1^*.$$

Similarly, there exist complex numbers γ and δ such that

$$\phi(zy^*) = \gamma x_1w_1^* + \delta z_1y_1^*.$$

Let us now introduce operators $P = (1/2)(x - z)(y - w)^*$ and $N = (x - z)(y + w)^*$. It is easy to verify that $P + \lambda N$ is an idempotent for every $\lambda \in \mathbb{C}$. As before, we see that there exists an integer $n \geq 2$ such that

$$(\phi(P) + \lambda\phi(N))^n = \phi(P) + \lambda\phi(N)$$

for every $\lambda \in \mathbb{C}$. The coefficient at λ^n must be zero, and consequently,

$$\phi(N) = \mu_1x_1y_1^* - \mu_2z_1w_1^* + (\alpha - \gamma)x_1w_1^* + (\beta - \delta)z_1y_1^*$$

is a nilpotent operator. The linear span of the set $\{x_1y_1^*, z_1w_1^*, x_1w_1^*, z_1y_1^*\}$ is isomorphic to the set of all 2×2 matrices via the isomorphism

$$a_1x_1y_1^* + a_2x_1w_1^* + a_3z_1y_1^* + a_4z_1w_1^* \mapsto \begin{Vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{Vmatrix}.$$

It is now easy to see that the fact that $\phi(N)$ is nilpotent is in a contradiction with $\mu_1 \neq \mu_2$.

Thus, we have proved that $\phi(I) = cI$ for some root of unity c. The bijective linear mapping $\varphi\colon \mathscr{B}(H) \to \mathscr{B}(K)$ defined by $\varphi(A) = \bar{c}\phi(A)$ preserves potents in both directions. Moreover, we have that $\varphi(I) = I$. If P is an arbitrary idempotent from $\mathscr{B}(H)$, then P, $I - P$, and $I - 2P$ are potent operators. The same must be true for $\phi(P)$, $I - \phi(P)$, and $I - 2\phi(P)$. This implies that $\phi(P)$ is an idempotent. An application of Theorem 3 now concludes the proof of the implication (i) $\Rightarrow$ (iii).

Assume now that there exists an integer $r \geq 2$ such that ϕ preserves r-potent operators in both directions. As before we see that ϕ must be injective. Let P, $Q \in \mathscr{B}(H)$ be idempotents such that $PQ = QP = 0$. The same proof as in the finite-dimensional case [3] shows that $\phi(P)\phi(Q) = \phi(Q)\phi(P) = 0$. Moreover, we have $\phi(P)^r = \phi(P)$ and $\phi(Q)^r = \phi(Q)$, and consequently,

$$\phi((\lambda P + \mu Q)^r) = \lambda^r \phi(P) + \mu^r \phi(Q) = (\phi(\lambda P + \mu Q))^r$$

for arbitrary λ, $\mu \in \mathbb{C}$.

Let P be an idempotent from $\mathscr{B}(H)$. Then we have for every complex number λ that

$$\phi((P + \lambda I)^r) = \phi(((1+\lambda)P + \lambda(I - P))^r) = (\phi(P) + \lambda\phi(I))^r .$$

Comparing the coefficients at λ^{r-1} we obtain that

$$r\phi(P) = \phi(P)S^{r-1} + S\phi(P)S^{r-2} + \cdots + S^{r-1}\phi(P),$$

where $S = \phi(I)$. Multiplying this relation first from the left by S, then from the right by S, and using $S^r = S$ we get that $\phi(P)S = S\phi(P)$. Theorem 1 and surjectively of ϕ imply that S belongs to the center of $\mathscr{B}(K)$. Applying $S^r = S$ once again, we conclude that $\phi(I) = cI$ for some $(r-1)$-root of unity.

The bijective linear mapping $\varphi : \mathscr{B}(H) \to \mathscr{B}(K)$ given by $\varphi(A) = \overline{c}\phi(A)$ preserves r-potent operators in both directions and satisfies $\varphi(I) = I$. We shall conclude the proof of implication (ii) $\Rightarrow$ (iii) and therefore the theorem by showing that it preserves idempotents. Let P be an idempotent operator on H. Then we have $\varphi(P)^r = \varphi(P)$ and $(I - \varphi(P))^r = I - \varphi(P)$. Assume that $\sigma(\varphi(P))$ contains a complex number $\lambda \notin \{0, 1\}$. From $|\lambda| = |1 - \lambda| = 1$ it follows that either λ or $\overline{\lambda}$ is equal to $\exp((1/3)\pi i)$, which is in contradiction with $\varphi(P)^r = \varphi(P)$ in the case that 6 is not a divisor of $r - 1$. In the case that 6 divides $r - 1$ one can easily verify that $I + \exp((2/3)\pi i)P$ and $I + \exp((4/3)\pi i)P$ are r-potent operators. The same must be true for their φ-images. It follows that $\sigma(\varphi(P)) \subset \{0, 1\}$, or in other words, $\varphi(P)$ is an idempotent. This completes the proof.

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Department of Mathematics, University of Ljubljana, Jadranska 19, 61000 Ljubljana, Slovenia

E-mail address: matjaz.omladic@uni-lj.si

E-mail address: peter.semrl@uni-lj.si